KINETIC MODELING AND THE SOCIAL IMPACT OF PUBLIC POLICY IN CONTROLLING DISEASE SPREADING.

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Abstract. We study the propagation of a disease in a population where agents are characterized by their awareness level, representing the measures they take to avoid the infection. We introduce another agent, the government, which is constantly sending a message to the population trying to steer the mean awareness to a value which should ensure the extinction of the disease. We propose three levels to analyze this model. First, an agent-based model, which we use later to derive a mean-field system of ordinary differential equations; and finally, we propose a kinetic approach to model the evolution of the distribution of agents on the awareness levels. We obtain a nonlinear ODEs-PDE system, where a first order, non-local-partial differential equation is coupled with two ordinary differential equations that describe the evolution of the epidemic and the response of the government. We prove the existence and uniqueness of solutions.

1. Introduction

When affected by a new disease, a population can adopt atypical social behaviours, like social distancing, lock-downs, school closures, and many other measures in order to control the spread of the disease. This is a well-documented phenomena, that are we experiencing now, and can be traced back at least to the *Plague of Athens* described by Thucydides. Those behavioral changes affect the dynamic of the disease, which in turn induces changes on the social habits of the population. The study of the disease dynamic must take into account the social response of the population, giving rise to an irreducible complex human-disease, according to [5].

However, classical mathematical models of diseases dynamics, like the seminal work of Kerman-MacKendrick in the *Susceptible - Infected - Recovered* (SIR) model and its numerous variants, usually neglect the social aspect of the problem. Up to the authors knowledge, the first paper taking into account explicitly the social impact of disease control measures seems to be [10]. Since then, increasing attention has been given to the modelling of the complex *human-disease*, resulting in a new field of research named *behavioral epidemiology*, we refer to the survey [6] for more information, and see also [29].

During the Covid-19 pandemic, in the absence of a vaccine, most governments relied on policies that aimed at reducing contacts between individuals, and at raising hygiene levels by the so-called barrier gestures so as to lower the infection rate. Thus, it is important to model such government health policies and their impact on the population to predict the future dynamics of the disease and evaluate their efficiency. In fact, this issue is one the nine challenges in behavioral epidemiology identified in [16].

Key words and phrases. epidemiology, kinetic equations, SIS models, opinion dynamics

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Disease dynamic is traditionally studied by dividing the population in compartments. The evolution of the proportion of individuals in each compartment is then given by a a system of ordinary differential equations (ODEs) that can be studied numerically and theoretically by using tools from dynamical systems theory. The most well-known compartments models are the SIR and SIS, *Susceptible - Infected -Susceptible* models. In this framework the impact of media or government policy is incorporated either by assuming that the transition rates between compartments (in particular, the contact rate) depends on the proportion of infected individuals, or by adding new compartments for the *aware* individuals. Also, when the media or the government propagates a signal, the frequency or intensity of their announces is obtained from an extra ordinary differential equation, which increases when the number of cases rises or people does not obey the instructions, and is constrained by budget restrictions and the capacity of attention of the individuals, see [20], and also [28] for a recent survey.

This approach has proven useful but has some drawbacks: first, the population is assumed to be homogeneous, and second, it allows only for a crude coarse-grained approach of the social dynamic since only a few compartments are used to model the social heterogeneity.

On the other hand, the computational power available nowadays allows first to consider models which keep track of the state of each individual at any time, and also to model interactions (both infections and social processes) on a microscopic level [9, 12, 15, 30].

In that setting an individual is usually characterized by its state with respect to the disease (susceptible or infected), and by another parameter, its awareness, quantifying its knowledge of the disease and the measures taken to avoid contagion. Interactions between agents lead to a co-evolution of the disease status and the awareness, in agreement with the *human disease* paradigm. These Agents-Based Models (ABM) are thus very realistic but also computationally intensive. In the particular case where any pair of agents can interact, it is possible to obtain a mean-field description of the dynamic with a system of ordinary differential equations, as obtained recently in [18].

The evolution of the (individual) awareness parameter is governed by social interactions and also by the dynamics of the disease, and its modelling can thus take advantage of the vast literature concerning the opinion formation process. From a mathematical point of view, a kinetic approach has been proven useful in studying the time evolution of the distribution of opinions in a population (see e.g. [2, 21, 22, 23, 26, 27] and references therein). In that framework the distribution of agents is a probability measure over the space of available opinions that evolves following a Boltzmann type equation. In some cases, its time evolution can be approximated by a Fokker-Planck equation. In particular, the presence of stubborn agents that refuse to implement protection measures can be studied as in [24].

Kinetic equations were used in the recent paper [11] where a SIR models is developed considering that each individual is characterized by its contact rate. Thus, the distribution of individual contact rates in the susceptible, infected and recovered subpopulation satisfies a system of partial differential equations. However, the contact rates remained unchanged. Also, in [1, 7] the authors used the kinetic approach of active particles to model the heterogeneity of agents respect to viral load and protection measures among others.

In our work we study the propagation of a disease in a population where individuals are willing to adjust their awareness level following the recommendation of a government institution, and the government in turn constantly sends a public message to the population aiming at an awareness value which should be enough to obtain the extinction of the epidemic. In Section §2 we present the precise rules of interaction, we derive the finite-dimensional mean field equations, and we study the stability of the equilibria. Section §3 is devoted to the kinetic model and we get a Boltzmann type equation coupled with the SIS model and the government adjustment of the signal. We prove the existence and uniqueness of solutions by following the ideas of Bressan [8], see also [3], by considering the system as an abstract ordinary differential equation in a subset of a Banach space. Moreover, we perform the so-called *grazing limit* and we obtain an ODEs-PDE system, by replacing the Boltzmann equation by a first order, non-local partial differential equation for the distribution of agents in the awareness levels. Also, we add stubborn agents, and we analyze their impact on the existence of an endemic equilibria where the disease persists. The proof of the main theorems in Section §3 are given in Appendix A to D, for the ease of the presentation, due to the technical details involved. Finally, in Section §4 we conclude the paper and analyze possible extensions.

2. Description of the model and analysis of the mean-field equations.

2.1. **Description of the model.** We consider a population of N agents in which the spreading of a disease, modelled by a standard SIS model. We denote α and β the contagion and recovery rates. Thus, contagion and recovery occur following a Poisson process with rates α and β. It is well known that in a large well-mixed population the mean proportion I of infected agents satisfies the ordinary differential equation

$$\frac{d}{d\tau}I = \alpha I(1-I) - \beta I. \tag{2.1}$$

We modify this model assuming that each individual is characterized by a level of awareness $a \geq 0$ modelling its knowledge about prevention measures against the disease. An individual with awareness $a \simeq 0$ has a high probability of getting infected after contact with an infected individual. On the other hand an individual with high awareness $a \gg 0$ takes almost all the necessary precautions to avoid contagion. We model this intuition in the following way. A healthy individual with awareness a interacts in a time interval dt with an infected agent and becomes infected with probability $\alpha e^{-a}dt$. We denote $a_1, \dots, a_N$ the awareness level of each of the N individuals in the population. We will see below that the mean proportion I of infected agents satisfies the equation

$$\frac{d}{d\tau}I = \alpha\langle e^{-a}\rangle I(1-I) - \beta I, \tag{2.2}$$

where $\langle e^{-a}\rangle = \frac{1}{N}\sum_{i=1}^{N} e^{-a_i}$ is the mean value of e^{-a} in the population at time t.

We suppose that the awareness levels of the agents can change in time due to the influence of public policy. We assume that a government is continuously sending a message $W(t)$ to the whole population to drive the individual awareness level $a_i(t)$, $i = 1, \dots, N$, to some level a^* chosen to ensure the extinction of the disease. To determine a^*, recall that in the basic SIS model we have $\lim_{t\to+\infty} I(t) = 0$ if and only if the basic reproduction number $R_0 := \alpha/\beta$ is less than or equal to 1. If all individuals have awareness level $a_i = a^*$, $i = 1, .., N$, then the basic reproduction number for (2.2) is $\alpha e^{-a^*}/\beta$. The threshold for the extinction of the disease is then $\alpha e^{-a^*}/\beta = 1$, i.e.

$$a^* = \log\left(\frac{\alpha}{\beta}\right). \tag{2.3}$$

From a practical point of view it will be more convenient for the government to try to drive the population's awareness to a value strictly greater than a^* when $\alpha > \beta$, say $a^*_\delta := a^* + \delta$ for some $\delta > 0$. When $\alpha < \beta$, we will take δ so that $a^*_\delta = a^*$.

We model the evolution of the government message $W(t)$ with the equation

$$W'(t) = W(t)\Big(\rho I(t) + \sigma(a^*_\delta - \langle a\rangle)_t + \eta(a^*_\delta - W(t))\Big), \tag{2.4}$$

where $\rho, \sigma, \eta, \delta > 0$ are fixed positive parameters, and $\langle a\rangle_t = \frac{1}{N}\sum_{i=1}^N a_i(t)$ is the mean value of awareness in the population at time t. Observe that the rate of change of the signal strength grows proportionally to three factors: the proportion of infected agents $\rho I(t)$, the difference between the desired value a^*_δ and the mean awareness level in the population $\langle a\rangle_t$, and the difference between the desired value a^*_δ and the current government message $W(t)$.

We assume that interactions with the government occur with unit rate and that, upon interaction, each individual i is willing to comply to the government recommendation in the sense that it will modify its awareness level to

$$a_i(t + dt) = a_i(t) + \gamma dt(W(t) - a_i(t))$$

where $\gamma > 0$ models the strength of the commitment of the individuals to comply. Notice that, when $a_i > W$, then a_i decreases thus getting closer to W.

The complete agent-based dynamic is given by the following algorithm where we denote $s_i \in \{0, 1\}$ the status of agent i ($s_i = 1$ if infected, $s_i = 0$ if not).

Algorithm 1: Agent based dynamic

```
Data: time step dτ, number of time steps N_τ,
      modelling parameters α, β, δ, γ, ρ, σ, η.
Result: a_1, ..., a_N, s_1, ..., s_N, W at time k dτ, k = 0, ..., N_τ.
initialization of a_1, ..., a_N, s_1, ..., s_N, W.
for τ ← 1 to N_τ do
    for k ← 1 to N do
        select an agent i at random.
            a'_i = a_i + γ dτ (W − a_i);
        select two distinct agents i and j at random.
        if s_i = 0 and s_j = 1 and rand < α e^{−a_i} dτ then
            s'_i = 1
        select an agent i at random.
        if s_i = 1 and rand < β dτ then
            s'_i = 0;
        Update.
    Save a_1, ..., a_N, s_1, ..., s_N.
    Compute ⟨a⟩ = (1/N) Σ_i a_i, I = (1/N) Σ_i s_i, and
    W ← W + dτ W (ρI + σ(a*_δ − ⟨a⟩) + η(a*_δ − W))
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Here, *rand* denotes a number drawn at random from the uniform distribution in $[0, 1]$.

We show in Figure 1 the time evolution of the proportion $I(t)$ of infected agents, the government message $W(t)$, the mean awareness level $\langle a\rangle = \frac{1}{N}\sum_i a_i$, and a^*_δ, using the agent

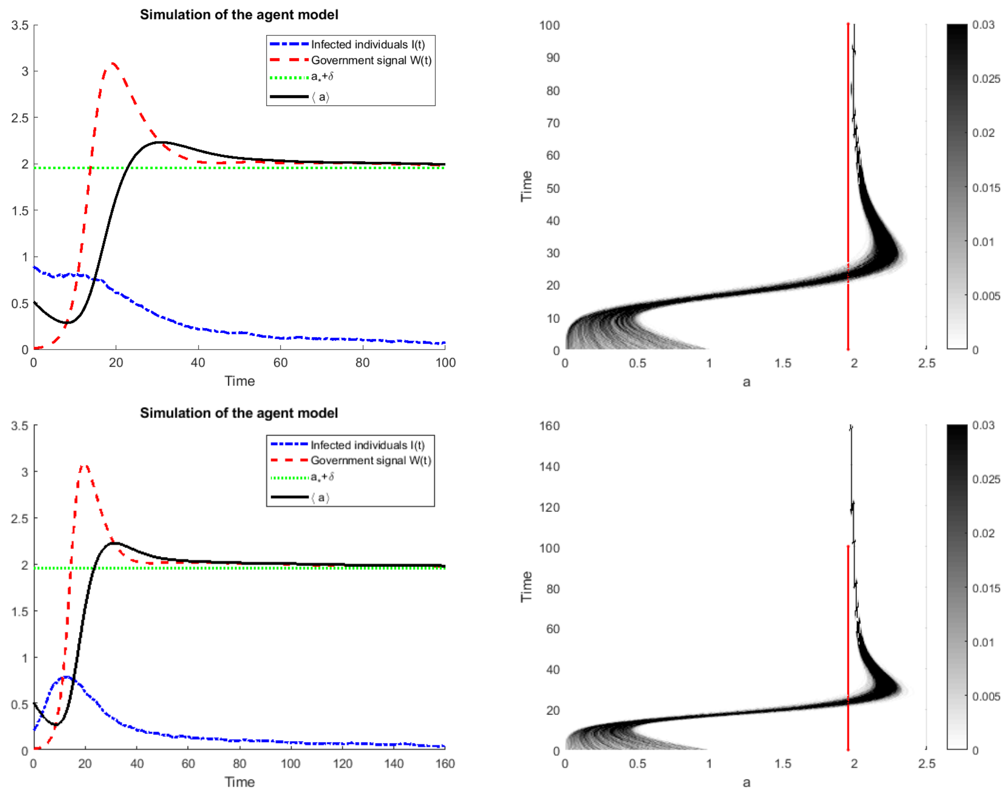

FIGURE 1. Left panel: Time evolution of $I(t)$, $W(t)$, and $\langle a\rangle = \frac{1}{N}\sum_i a_i$, using Algorithm 1 with parameters (2.5). Right panel: Time evolution of the awareness levels $a_1, \ldots, a_N$; the vertical red line is $a = a^*_\delta$. Top row: initial proportion of infected agents 0.9, bottom row: 0.2

based model described in Algorithm 1 with parameters (2.5):

$$\alpha = 0.7, \quad \beta = 0.1, \qquad \gamma = \rho = \sigma = \eta = 0.1, \qquad d\tau = 0.1, \qquad \delta = 0.01. \tag{2.5}$$

In the right panel we plot the time evolution of the distribution of awareness levels $a_1, \ldots, a_N$. The vertical red line is $a = a^*_\delta$.

We initialize the dynamics as follows: the initial government message is $W(0) = 0.01$, and agents awareness levels are randomly chosen with an uniform distribution in $[0, 1]$. We considered two initial proportions of infected agents: 0.9 (top row) and 0.2 (bottom row). Notice that since $\alpha > \beta$, the epidemic persists in the clasical SIS model. However, due to the action of the government, we can see that the disease goes to extinction in our model.

2.2. **Mean-field approximation for a large and well-mixed population.** We now want to obtain the differential equations governing the evolution of the proportion I of infected agents, the government message W and the awareness levels $a_1, \ldots, a_N$ following the algorithm 1.

The evolution of the government message is given by

$$\frac{d}{d\tau}W = W(\rho I + \sigma(a^*_\delta - \langle a\rangle) + \eta(a^*_\delta - W)).$$

Since any agent i has a probability $\frac{1}{N}$ to be involved in an interaction, we have for $dt \ll 1$ that

$$a_i(t+dt) \simeq a_i(t) + \frac{dt}{N}\gamma(W(t) - a_i(t)). \tag{2.6}$$

Also, in small time intervals $[t, t+dt]$ there is a contagion with probability

$$\sum_{i\,susceptible} \frac{1}{N} \sum_{j\,infected} \frac{1}{N-1}\alpha e^{-a_i(t)}dt = \alpha dt I(t)\frac{1}{N}\sum_{i\,susceptible} e^{-a_i(t)}.$$

We denote N_S the number of susceptible agents, so that $N_S/N = 1 - I$, and the last sum is

$$(1-I)\frac{1}{N_S}\sum_{i\,susceptible} e^{-a_i(t)}.$$

Thus, the contagion probability is approximately

$$\alpha dt I(t)(1-I(t))\frac{1}{N_S}\sum_{i\,susceptible} e^{-a_i(t)}.$$

On the other hand, there is a recovery with probability

$$\sum_{i\,infected} \frac{1}{N}\beta dt = I(t)\beta dt.$$

Thus,

$$I(t+dt) \simeq I(t) + \frac{dt}{N}\Big\{\alpha I(t)(1-I(t))\frac{1}{N_S}\sum_{i\,susceptible} e^{-a_i(t)} - \beta I(t)\Big\}. \tag{2.7}$$

Notice that the random variables

$$\langle e^{-a}\rangle^{Sus}_{t=0} := \frac{1}{N_S}\sum_{i\,susceptible} e^{-a_i(0)}, \qquad \langle e^{-a}\rangle_{t=0} := \frac{1}{N}\sum_{i=1..N} e^{-a_i(0)}$$

have the same distribution since the individual levels $a_1(0), \dots, a_N(0)$ are independent and identically distributed. Moreover, the updating rule of a_i does not distinguish between infected and susceptible agents. Since equation (2.7) holds in the mean (i.e., averaging over many realizations of the dynamic), it is intuitively reasonable to replace $\langle e^{-a}\rangle^{Sus}$ by $\langle e^{-a}\rangle$ thus obtaining

$$I(t+dt) \simeq I(t) + \frac{dt}{N}\Big\{\alpha\langle e^{-a}\rangle I(t)(1-I(t)) - \beta I(t)\Big\} \tag{2.8}$$

which is Equation (2.2), the one we used to determine a^*. We show in Figure 2 the time evolution of $\log(|\langle e^{-a}\rangle^{Sus} - \langle e^{-a}\rangle|)$ during a run of the agent simulation. We can appreciate that $\langle e^{-a}\rangle^{Sus} \simeq \langle e^{-a}\rangle$.

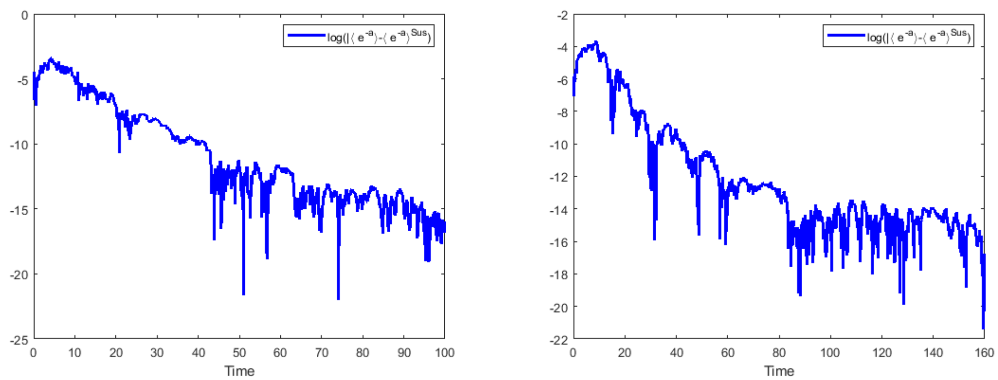

FIGURE 2. Time evolution of the logarithm of the distance between $\langle e^{-a}\rangle$ and $\langle e^{-a}\rangle^{Sus}$ in a simulation of the agent-based model. We observe that this value reaches the machine epsilon precision.

In view of the factor $\frac{dt}{N}$ in equations (2.6) and (2.8), it is natural to rescale time considering $\tau := t/N$ so that $d\tau = \frac{dt}{N}$ (this explains the notation τ used in Algorithm 1). In the limit $d\tau \to 0$, we obtain that $a_1, \ldots, a_N, I, W$ verify the system of ordinary differential equations

$$\begin{aligned} &\frac{d}{d\tau} I = \langle e^{-a}\rangle \alpha I(1-I) - \beta I, \\ &\frac{d}{d\tau} a_i = \gamma(W - a_i) \qquad i = 1, .., N, \\ &\frac{d}{d\tau} W = W\Big(\rho I + \sigma(a^*_\delta - \langle a\rangle) + \eta(a^*_\delta - W)\Big). \end{aligned} \tag{2.9}$$

We show in Figure 3 the numerical resolution of this system for the same parameters and initial condition as those used in the simulation of in Figure 1. Notice that both figures are nearly indistinguishable confirming that the mean-field system (2.9) adequately captures the evolution of the agent-based model.

2.3. Analytical results.

From now on we denote t for τ. Consider some initial condition $a_1(0), .., a_N(0) \geq 0$, $W(0) \geq 0$, and $I(0) \in [0, 1]$. Then system (2.9) has a unique solution $a_1(t), \ldots, a_N(t), W(t), I(t)$ defined in an open maximal interval $J \ni 0$. Moreover, it is easily seen that $W(t) \geq 0$ and $I(t) \in [0, 1]$ for any $t \in J$. Also, from

$$\frac{d}{d\tau} W \leq \eta W\Big(A - W\Big), \qquad A = \frac{\rho}{\eta} + (1 + \frac{\sigma}{\eta})a^*_\delta$$

we see that that $W(t) \leq \max\{W(0), A\}$. Thus the solution $(a_1(t), \ldots, a_N(t), W(t), I(t))$ remains bounded so that $J = \mathbb{R}$, i.e. the solution exists for all $t \in \mathbb{R}$.

We are interested in the long-time behaviour $t \to +\infty$ of $a_1(t), \ldots, a_N(t), W(t), I(t)$.

The case $\alpha < \beta$ is simple since I can be controlled by the standard SIS model and thus $I(t) \to 0$. No govermnent intervention is needed: $W(t) \to 0$, and also $a_i(t) \to 0$.

Teorem 2.1. *Assume that $\alpha < \beta$. Then as $t \to +\infty$,*

$$a_1(t), \ldots, a_N(t) \to 0, \qquad W(t) \to 0, \qquad I(t) \to 0.$$

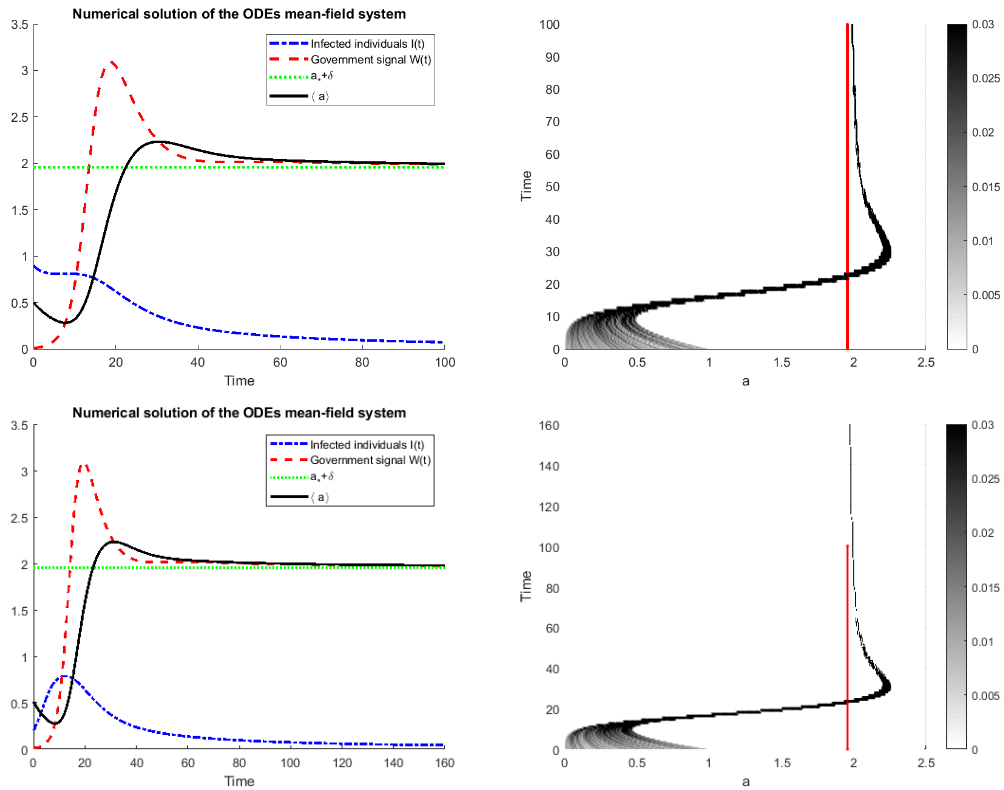

FIGURE 3. Numerical solution of the system (2.9) with the same parameters and initial condition as in Figure 1.

Proof. Since $\langle e^{-a}\rangle_t^S \leq 1$, we have

$$I'(t) \leq \alpha I(t)(1 - I(t)) - \beta I(t),$$

so that $I(t) \leq I_{sis}(t)$ where I_{sis} solves the standard SIS model:

$$I'_{sis}(t) = \alpha I_{sis}(1 - I_{sis}) - \beta I_{sis}, \qquad I_{sis}(0) = I(0).$$

Since $\alpha < \beta$, $I_{sis}(t) \to 0$ and then $I(t) \leq I_{sis}(t) \to 0$.

Letting $\varepsilon(t) := \rho I(t)$ and recalling that $a^*_\delta = a^* < 0$ (because $\alpha < \beta$), we have

$$W'(t) \leq W(t)(\varepsilon(t) - \gamma W(t)).$$

It follows that $W(t) \to 0$. Otherwise there would exist $\delta > 0$ and a sequence $t_k \uparrow +\infty$ such that $W'(t_k) = 0$ and $W(t_k) \geq \delta$. Then

$$0 = W'(t_k) \leq W(t_k)(\varepsilon(t_k) - \gamma W(t_k)) \leq W(t_k)(\varepsilon(t_k) - \gamma\delta)$$

so that $0 \leq \varepsilon(t_k) - \gamma\delta$ and the right hand side is negative for $k \gg 1$, a contradiction. A similar argument shows that $a_i(t) \to 0$, $i = 1, \ldots, N$, and the proof is finished. □

The case $\alpha > \beta$ is the most interesting one, since the government must act to control the spreading of the disease. We start its analysis by identifying the equilibria.

Teorem 2.2. *For any $\alpha > \beta$ and $\delta > 0$, the equilibria $(I, a_1, \dots, a_N, W)$ of (2.9) are the disease free equilibrium $(0, \dots, 0)$ and $(0, a^*_\delta, \dots, a^*_\delta)$, and the equilibrium $(1 - \frac{\beta}{\alpha}, 0, \dots, 0)$.*

Proof. The proof is straightforward. From (2.9) we have $a_1 = \dots = a_N = W$, so that it remains to solve

$$I\Big(1 - \frac{\beta}{\alpha}e^W - I\Big) = 0, \qquad W\Big(\rho I + (\sigma+\eta)(a^*_\delta - W)\Big) = 0. \tag{2.10}$$

If $W = 0$ we obtain $I = 0$ or $I = 1 - \frac{\beta}{\alpha}$, thus giving the equilibria $(0, .., 0)$ and $(1 - \frac{\beta}{\alpha}, 0, ..., 0)$.

If $W \neq 0$ then $\rho I + (\sigma + \eta)(a^*_\delta - W) = 0$. If $I = 0$, we obtain the third equilibrium $(0, a^*_\delta, ..., a^*_\delta)$. If $I \neq 0$, we obtain the system (2.11). If $\delta \geq 0$, this system has no solution. Indeed, since $I > 0$ we must have $a^*_\delta - W < 0$, i.e. $e^W > e^\delta \frac{\alpha}{\beta}$, so that $1 > 1 - I = \frac{\beta}{\alpha}e^W > e^\delta$ which implies $\delta < 0$. The proof is finished. □

In view of the numerical simulations shown in the previous section, we conjecture that, when $\delta > 0$, then any solution with $W(0) > 0$ converges as $t \to \infty$ to the disease-free equilibrium $P := (0, a^*_\delta, \dots, a^*_\delta, a^*_\delta)$. A linear stability analysis shows that P is the only equilibrium which is locally asymptotically stable when $\delta > 0$. We still suppose that $\alpha > \beta$.

Teorem 2.3. *The equilibria $(0, ..., 0)$ and $(1 - \frac{\beta}{\alpha}, 0, \dots, 0)$ are unstable. Moreover, by denoting A the linearized matrix of (2.9) around the disease-free equilibrium $P := (0, a^*_\delta, \dots, a^*_\delta, a^*_\delta)$, then there exists a constant $C > 0$ independent of N such that for any N all the eigenvalues of A belong to $\{z \in \mathbb{C} : Re(z) < -C\}$. In particular, P is locally asymptotically stable for any $\delta > 0$ and any $N \geq 1$.*

Proof. See Appendix A. □

Remark 2.1. *When $\delta < 0$, the equilibria of the system are the three equilibria of Theorem 2.2, which are unstable, and a fourth equilibrium $(i^*, w^*, ..., w^*)$ (existing only if $\delta < 0$) where (i^*, w^*) is the unique solution of the system*

$$1 - \frac{\beta}{\alpha}e^W = I, \qquad \rho I + (\sigma+\eta)(a^*_\delta - W) = 0. \tag{2.11}$$

Indeed, if $\delta < 0$, system (2.11) has a unique solution since the function

$$f(i) := \rho i + (\sigma + \eta)\left(a^*_\delta - \ln\left(\frac{\alpha}{\beta}(1-i)\right)\right)$$

is increasing with $f(0) < 0$ and $f(1 - \beta/\alpha) > 0$. Thus there exists a unique i^* such that $f(i^*) = 0$ and $i^* \in (0, 1 - \beta/\alpha)$, in particular $w^* = \ln\left(\frac{\alpha}{\beta}(1-i)\right) > 0$. Notice that $(i^*, w^*) \in (0, 1 - \beta/\alpha) \times (0, +\infty)$.

We conjecture that this equilibrium is asymptotically stable as illustrated in Figure 4 which displays a run of the dynamic with parameters (2.5) being $\delta = -0.01$, and $I(0) = 0.2$, $W(0) = 0.01$.

Recall that $1 - \beta/\alpha$ is the asymptotic proportion of infected individuals in the endemic state of the classical SIS model. Since $i^* < 1 - \beta/\alpha$ we see that the government action succeeds in lowering the limit proportion of infected individuals as compared with the SIS model.

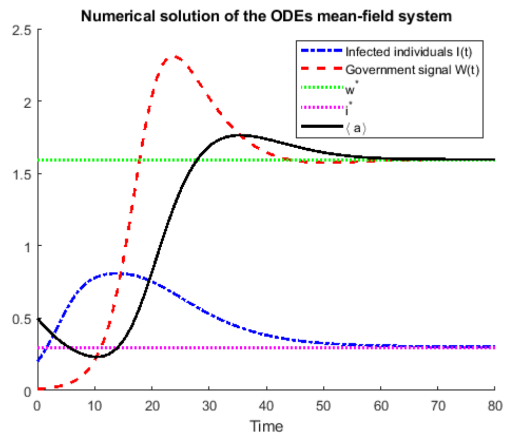

FIGURE 4. Numerical resolution of the system 3 with same parameters and initial condition as in Figure 1, with $\delta = -0.5$. The horizontal lines correspond to (i^*, w^*) of system (2.11).

If we suppose that the social dynamic is much faster than the disease dynamic, i.e. $\gamma \gg 1$, then all agents follow quickly the government directions. Formally $a_i = W$ for all $i = 1, \dots, N$, and we can reduce system (2.9) to:

$$\begin{aligned} \frac{d}{d\tau} I &= \langle e^{-W} \rangle \alpha I(1-I) - \beta I, \\ \frac{d}{d\tau} W &= W\Big(\rho I + (\sigma+\eta)(a_\delta^* - W)\Big). \end{aligned} \tag{2.12}$$

We can prove now:

Proposition 2.4. *For any initial condition (W_0, I_0) with $W_0 > 0$ and $I_0 \in (0,1]$, the corresponding solution $(W(t), I(t))$ of (2.12) converges to $(a_*^\delta, 0)$ as $\tau \to +\infty$.*

Proof. We divide the quadrant $Q := \{(W,I),\ W > 0,\ I > 0\}$ in the three zones I, II and III defined by

$$\begin{aligned} I &= \{(W,I) \in Q :\ \langle e^{-W} \rangle \alpha(1-I) - \beta > 0\}, \\ III &= \{(W,I) \in Q :\ \rho I + (\sigma+\eta)(a_\delta^* - W) < 0\}, \\ II &= Q \backslash (I \cup III). \end{aligned}$$

Notice that the vector-field defining the right hand side of (2.12) points inside III in the line $\{(W,I) \in Q :\ \rho I + (\sigma+\eta)(a_\delta^* - W) = 0\}$ so that III is invariant. In III, a solution $(W(t), I(t))$ verifies $I', W' < 0$ with $I > 0$, $W > a_\delta^*$, so that the limit $\lim_{t\to+\infty}(W(t), I(t))$ must exists. This limit must be an equilibrium belonging to III, i.e. $(a_\delta^*, 0)$.

It is easily seen that if a solution starts from I, it must pass to II in finite time. If then it passes to III then we are done. Otherwise, if it stays in II for $t \gg 1$, then since $I' < 0$ and $W' > 0$ it must converges to the unique equilibrium in II which is $(a_\delta^*, 0)$. The proof is finished. □

3. The kinetic approach and the influence of stubborn agents.

Up to now we supposed that all agents were willing to comply with the government recommendations. However it is well-known that not all individuals respond in the same way to public policy as the public demonstrations of opposition to quarantine and barrier gestures in various countries. To model this, we assume that besides its awareness level a each individual is also characterized by an additional parameter $q \in [0,1]$ that models its volatility or willingness to change its mind. When interacting with the government at time t an agent will now modify its awareness level a as

$$a \to a + q\gamma(W(t) - a) \tag{3.1}$$

Notice that individual with $q = 0$ are stubborn or zealots in the sense that they do not follow the government recommendation at all.

Moreover, individuals communicate their opinions, i.e. their awareness level, through interactions with family, friends, colleagues, or neighbors. We assume that these interactions are binary and occur at the same unit rate as interactions with the government. We also suppose that the parameter q is not modified during interactions. When an agent with parameters (a, q) interacts with another agent with parameter (a_*, q_*), it will modify only its awareness level ending up with a post-interaction awareness level a'. We assume that

$$a' = a + q\kappa(a_* - a), \tag{3.2}$$

where $\kappa > 0$. Notice once again that stubborn individuals, those that have $q = 0$, do not change their awareness level.

We can deduce the corresponding mean-field equations as before modifying (2.6) as

$$\begin{aligned} a_i(t+dt) &\simeq a_i(t) + \frac{dt}{N} q_i \Big\{ \gamma(W(t) - a_i(t)) + \sum_j \frac{1}{N-1} \kappa(a_j(t) - a_i(t)) \Big\} \\ &\simeq a_i(t) + \frac{dt}{N} q_i \Big\{ \gamma(W(t) - a_i(t)) + \kappa(\langle a \rangle_t - a_i(t)) \Big\}. \end{aligned} \tag{3.3}$$

Equations for I and W are the same before. We thus obtain the system

$$\begin{aligned} &\frac{d}{d\tau} I = \langle e^{-a} \rangle \alpha I(1-I) - \beta I, \\ &\frac{d}{d\tau} a_i = q_i \Big\{ \gamma(W(t) - a_i(t)) + \kappa(\langle a \rangle_t - a_i(t)) \Big\} \qquad i = 1, .., N, \\ &\frac{d}{d\tau} W = W\Big(\rho I + \sigma(a^*_\delta - \langle a \rangle) + \eta(a^*_\delta - W) \Big). \end{aligned} \tag{3.4}$$

To get a better understanding of of system (3.4) it is useful to rewrite it in a more compact way replacing the N equations for $\frac{d}{d\tau} a_i$ by a single equation. It is a common approach to model socio-economical phenomena with kinetic equations, see [21].

3.1. A kinetic approach.

In this section we propose a kinetic approach to model the interaction between agents between them and with the government. Denote f_t the distribution of the population in the space of parameters (a, q) at time t. Notice that f_t is a probability measure on $[0, +\infty) \times [0, 1]$. Given some function $\phi : [0, +\infty) \times [0, 1] \to \mathbb{R}$, the integral

$\int \phi(a,q)\, df_t(a,q)$ is the mean value of ϕ due to the population distribution at time t. For instance taking $\phi(a) = a$ this integral is

$$\int \phi(a)\, df_t(a) = \int a\, df_t(a) =: \langle a \rangle$$

the mean awareness level in the population at time t. Then, taking $\phi(a) = a^2 - \langle a \rangle^2$,

$$\int \phi(a)\, df_t(a) = \langle a^2 \rangle - \langle a^2 \rangle^2 =: Var(a)$$

is the variance of a.

It follows from the interaction rules (3.1) and (3.2) that the derivative of $\int \phi(a,q)\, df_t(a,q)$ is given by the following Boltzmann-like equation

$$\begin{aligned} \frac{d}{dt}\int \phi(a,q)\, df_t(a,q) = &\int [\phi(a + q\gamma(W(t) - a), q) - \phi(a,q)]\, df_t(a,q) \\ &+ \int [\phi(a + q\kappa(a_* - a), q) - \phi(a,q)]\, df_t(a,q) df_t(a_*, q_*) \end{aligned} \tag{3.5}$$

for any $\phi \in C_b([0,+\infty) \times [0,1])$.

We keep on modeling the action of the government $W(t)$ and the proportion $I(t)$ of ill people by (2.9). We thus obtain the system

$$\begin{aligned} \frac{d}{dt} I =& \langle e^{-a} \rangle \alpha I(1 - I) - \beta I, \\ \frac{d}{dt} W =& W\Big(\rho I + \sigma(a^*_\delta - \langle a \rangle) + \eta(a^*_\delta - W)\Big), \\ \frac{d}{dt}\int \phi(a,q)\, df_t(a,q) =& \int [\phi(a + q\gamma(W(t) - a), q) - \phi(a,q)]\, df_t(a,q) \\ &+ \int [\phi(a + q\kappa(a_* - a), q) - \phi(a,q)]\, df_t(a,q) df_t(a_*, q_*), \end{aligned} \tag{3.6}$$

where $\langle a \rangle = \int a\, df_t(a,q)$ and $\langle e^{-a} \rangle = \int e^{-a}\, df_t(a,q)$ are the mean-values of a and e^{-a} at time t. Notice that in the particular case $f_t = \frac{1}{N}\sum_{i=1}^N \delta_{(a_i,q_i)}$ then (3.6) reduces to (3.4).

We first prove the well-posedness of the coupled PDE - ODEs system (3.6). Given some initial conditions $f_0 \in P([0,+\infty) \times [0,1])$, $W_0 \geq 0$, $I_0 \in [0,1]$, a solution to (3.6) is a triple (f, W, I) with $f \in C([0,+\infty), P([0,+\infty) \times [0,1])) \cap C^1((0,+\infty), P([0,+\infty) \times [0,1]))$, $W \in C([0,+\infty), [0,+\infty))$, $I \in C([0,+\infty), [0,1])$ satisfying (3.6) for $t > 0$ and such that $f_{|t=0} = f_0$, $W(0) = W_0$, $I(0) = I_0$. Here $P([0,+\infty) \times [0,1])$ is endowed with the Bounded Lipschitz norm (see (B.1)).

We then have:

Teorem 3.1. *Consider initial conditions $f_0 \in P([0,+\infty) \times [0,1])$, $W_0 \geq 0$, $I_0 \in [0,1]$ where f_0 has compact support. Assume that $\gamma \in [0,1]$. Then there exists a unique solution to (3.6). Moreover there exists $R > 0$ independent of γ such that $supp\, f_t \subset [0,R] \times [0,1]$ for any $t \geq 0$.*

3.2. **Grazing limit.** The study of the long-time behaviour of (3.6) is a difficult problem mainly because equation (3.5) is non-local. A standard procedure known as quasi-invariant limit or grazing limit allow us to approximate the non-local equation (3.5) by a transport equation. To implement this idea, we rescale time and in the new time scale, all the parameters $\alpha, \beta, \gamma, \kappa, \rho, \sigma, \eta$ are multiplied by a small ε. We perform now a first order Taylor expansion in the equation of f_t in (3.6) resulting

$$\begin{aligned}\frac{1}{\varepsilon}\frac{d}{dt}\int \phi(a,q)\,df_t(a,q) \simeq& \gamma\int \partial_a\phi(a,q)q(W(t)-a)\,df_t(a,q)\\ &+\kappa\int \partial_a\phi(a,q)q(a_*-a)\,df_t(a,q)df_t(a_*,q_*)\\ =&\int \partial_a\phi(a,q)q\Big\{\gamma(W(t)-a)+\kappa(\langle a\rangle-a)\Big\}\,df_t(a,q).\end{aligned}$$

We thus obtain the approximating system

$$\begin{aligned}\frac{1}{\varepsilon}\frac{d}{dt}I =&\langle e^{-a}\rangle\alpha I(1-I)-\beta I,\\ \frac{1}{\varepsilon}\frac{d}{dt}W =&W\Big(\rho I+\sigma(a_\delta^*-\langle a\rangle)+\eta(a_\delta^*-W)\Big),\\ \frac{1}{\varepsilon}\frac{d}{dt}\int \phi(a,q)\,df_t(a,q) =& \int \partial_a\phi(a,q)q\Big\{\gamma(W(t)-a)+\kappa(\langle a\rangle-a)\Big\}\,df_t(a,q),\end{aligned}$$

for any $\phi \in C_b([0,+\infty)\times[0,1]$. By changing time considering $\tau := \varepsilon t$ and letting $g_\tau := f_t$, $W(\tau) := W(t)$, $I(\tau) = I(t)$ we obtain

$$\begin{aligned}\frac{d}{d\tau}I =&\langle e^{-a}\rangle\alpha I(1-I)-\beta I,\\ \frac{d}{d\tau}W =&W\Big(\rho I+\sigma(a_\delta^*-\langle a\rangle)+\eta(a_\delta^*-W)\Big),\\ \frac{d}{d\tau}\int \phi(a,q)\,dg_\tau(a,q) =& \int \partial_a\phi(a,q)q\Big\{\gamma(W(t)-a)+\kappa(\langle a\rangle-a)\Big\}\,dg_\tau(a,q),\end{aligned} \tag{3.7}$$

for any $\phi \in C_b([0,+\infty)\times[0,1]$. We thus expect to well approximate the long time behaviour of the solution of the original system (3.6) for small $\varepsilon > 0$.

Notice that the equation for g_τ is the weak form of the transport equation

$$\partial_\tau g_\tau + \partial_a\Big(q\Big\{\gamma(W-a)+\kappa(\langle a\rangle-a)\Big\}g_\tau\Big) = 0 \tag{3.8}$$

so that we can rewrite (3.7) in a compact way as

$$\begin{aligned}&\frac{d}{d\tau}I = \langle e^{-a}\rangle\alpha I(1-I)-\beta I,\\ &\frac{d}{d\tau}W = W\Big(\rho I+\sigma(a_\delta^*-\langle a\rangle)+\eta(a_\delta^*-W)\Big),\\ &\partial_\tau g_\tau + \partial_a\Big(q\Big\{\gamma(W-a)+\kappa(\langle a\rangle-a)\Big\}g_\tau\Big) = 0.\end{aligned} \tag{3.9}$$

Consider some initial conditions $(g_0, W(0), I(0)) \in P([0,+\infty)\times[0,1])\times[0,+\infty)\times[0,1]$. A solution in $[0,T^*)$ is a triple (g,W,I) with $g \in C([0,T^*), P([0,+\infty)\times[0,1])$, $W, I \in$

$C^1([0,T^*),\mathbb{R}^2)$ with $W(t)\geq 0$ and $0\leq I(t)\leq 1$ satisfying the initial condition and solving (3.7). The solution is global if $T^*=+\infty$.

The following result establishes the well-posedness of (3.7).

Teorem 3.2. *Consider some initial conditions* $(g_0, W(0), I(0)) \in P([0,+\infty)\times[0,1])\times[0,+\infty)\times[0,1]$ *and assume that* g_0 *has compact support in* $[0,+\infty)\times[0,1]$*. Then there exists a unique global solution* (g,W,I) *to* (3.7)*. Moreover there exists* $M, R_0>0$ *such that for any* $t\geq 0$*,* $W(t)\leq M$ *and* g_t *is supported in* $[0,R_0]\times[0,1]$*.*

We justify the approximation of (3.6) by (3.7):

Teorem 3.3. *Given initial conditions* $f_0\in P([0,+\infty)\times[0,1])$*,* $W_0\geq 0$*,* $I_0\in[0,1]$ *where* f_0 *has compact support, denote* $(f^\varepsilon, W^\varepsilon(t), I^\varepsilon(t))$ *the corresponding solution of* (3.6) *as given by Theorem 3.1. Let* $\tau=\varepsilon t$ *and* $g^\varepsilon\tau := f_t^\varepsilon$*. Then as* $\varepsilon\to 0$*, the following convergence holds for any* $T>0$*:*

$$\begin{aligned} &g^\varepsilon\to g \quad \textit{in } C([0,T],P([0,+\infty)\times[0,1])),\\ &W^\varepsilon\to W \quad \textit{in } C([0,T],[0,+\infty)),\\ &I^\varepsilon\to I \quad \textit{in } C([0,T],[0,1]),\end{aligned}$$

where (g,W,I) *is the unique solution of* (3.7) *with initial conditions* (f_0,W_0,I_0) *as given by Theorem 3.2.*

The proof of Theorem 3.1, Theorem 3.2 and Theorem 3.3 are given in Appendix B, C and D respectively.

3.3. **An approximate system.** Consider a solution (g,W,I) of system (3.7) as given by Theorem 3.2.

We first verify that the dynamic of the awareness distribution is contractive. What we mean by contractive is that conditioned to the value of q, the support of the awareness distribution $g_{\tau|q}$ shrinks to a single point. More precisely, denote $f_0(q)dq$ the distribution of the q parameter in the population, i.e. the projection of g_τ on $[0,1]$. Thanks to Jirina's theorem (see e.g. [19][III.5.9]) there exists a family $\{g_{\tau|q}\}_{q\in[0,1]}$ of probability measures over $[0,+\infty)$, unique in the complement of a set of zero measure foe $f_0(q)dq$, such that for any function $\phi:[0,+\infty)\times[0,1]\to\mathbb{R}$ integrable for g_τ,

$$\int \phi(a,q)\,dg_\tau(a,q)=\int_0^1\Big(\int_0^{+\infty}\phi(a,q)\,dg_{\tau|q}(a)\Big)\,f_0(q)dq.$$

We then have

Proposition 3.4. *For any* $q>0$ *in the support of* $f_0(q)dq$*,*

$$diam(conv(supp\, g_{\tau|q}))\leq diam(conv(supp\, g_{0|q}))e^{-q(\gamma+\kappa)\tau} \tag{3.10}$$

where $diam(conv(supp\, g_{\tau|q}))$ *is the diameter (or length) of the convex hull of the support of* $g_{\tau|q}$*.*

Proof. Slightly adapting the proof of [24][Step 3.9] to equation (3.8), it can be proved that $g_{\tau|q}$ satisfies

$$\partial_\tau g_{\tau|q}+q\partial_a\Big(\Big\{\gamma(W(\tau)-a)+\kappa(\langle a\rangle-a)\Big\}g_{\tau|q}\Big)=0. \tag{3.11}$$

Independently, let us recall that the cumulative distribution function $F_\tau : \mathbb{R} \to [0,1]$ of $g_{\tau|q}$, and its generalized inverse $X_\tau : [0,1] \to [0,+\infty)$ are defined by $F_\tau(x) = g_{\tau|q}((-\infty, x])$ and

$$X_\tau(\rho) = \inf\{x \in [0,+\infty) \text{ s.t. } F_\tau(x) \geq \rho\}, \qquad \rho \in [0,1]. \tag{3.12}$$

Since g_τ is supported in $[0,R_0]\times[0,1]$, $g_{\tau|q}$ is supported in $[0,R_0]$ so that $X_\tau(\rho) \in [0,R_0]$. Notice also that the segment $[X_\tau(0^+), X_\tau(1)]$ is $conv(supp\, g_{\tau|q})$, the convex hull of $g_{\tau|q}$. Eventually it is a classical property of X_τ that

$$\int_0^1 \phi(X_\tau(r))\, dr = \int_0^{+\infty} \phi(a)\, dg_{\tau|q}(a), \tag{3.13}$$

for any ϕ integrable (to prove this identity it suffices to check the formula for ϕ of the form $1_{(-\infty,a]}$, $a \in \mathbb{R}$). This change of variables allows us to rewrite equation (3.11) as (see e.g. [24][Prop. 3.1])

$$\partial_\tau X_\tau(r) = q\gamma(W - X_\tau(r)) + q\kappa(\langle a\rangle - X_\tau(r)) \qquad r \in (0,1).$$

It follows that for $0 < r < s < 1$,

$$\partial_\tau(X_\tau(s) - X_\tau(r)) = -q(\gamma+\kappa)(X_\tau(s) - X_\tau(r))$$

so that

$$X_\tau(s) - X_\tau(r) \leq (X_0(s) - X_0(r))e^{-q(\gamma+\kappa)\tau}.$$

Sending $s \to 1$ and $r \to 0^-$ gives (3.10). □

If we suppose that there exists $\varepsilon_0 \in (0,1)$ such that any non-stubborn agent has $q \geq \varepsilon_0$, i.e. $f^{NS}(q)dq$ is supported in $[\varepsilon_0, 1]$, then (3.10) gives

$$diam(conv(supp\, g_{\tau|q})) \leq diam(conv(supp\, g_{0|q}))e^{-\varepsilon_0(\gamma+\kappa)\tau} \tag{3.14}$$

for any q in the support of $f^{NS}(q)dq$. Thus each $g_{\tau|q}$ is very concentrated around its mean $a_\tau(q) = \int a\, dg_{\tau|q}(a)$. It is thus reasonable to replace, for $\tau \gg 1$ (independently of q), each $g_{\tau|q}$ by the Dirac mass $\delta_{a_\tau(q)}$. Notice that (3.11) gives

$$\frac{1}{q}\frac{d}{d\tau}a_\tau(q) = (\gamma(W(\tau) - a_\tau(q)) + \kappa(\langle a\rangle - a_\tau(q))$$

We will make a further simplification assuming that $a_\tau(q)$ is independent of q for $\tau \gg 1$, which is motivated by numerical simulations, i.e. $a_\tau(q) = a_\tau$. Then $\langle a\rangle^{NS} = a_\tau$. Recalling that $\langle a\rangle = \alpha_0\langle a\rangle^S + (1-\alpha_0)\langle a\rangle^{NS}$ we obtain

$$\frac{d}{d\tau}a_\tau = q\gamma(W(\tau) - a_\tau) + q\kappa\alpha_0(\langle a\rangle^S - \langle a\rangle^{NS}).$$

Integrating with respect to $f_0^{NS}(q)dq$ gives

$$\begin{aligned}\frac{d}{d\tau}\langle a\rangle^{NS} &= \langle q\rangle\Big(\gamma(W(\tau) - \langle a\rangle^{NS}) + \kappa\alpha_0(\langle a\rangle^S - \langle a\rangle^{NS})\Big)\\ &= \langle q\rangle(\gamma+\kappa\alpha_0)\Big(\frac{\gamma W(\tau) + \kappa\alpha_0\langle a\rangle^S}{\gamma+\kappa\alpha_0} - \langle a\rangle^{NS}\Big)\end{aligned} \tag{3.15}$$

Thus, the non-stubborn agents approximately react to a convex combination of the government signal and the opinion of the stubborn agents, which is precisely given by $\frac{\gamma W(\tau)+\kappa\alpha_0\langle a\rangle^S}{\gamma+\kappa\alpha_0}$.

With these approximations, system (3.9) becomes

$$
\begin{aligned}
&\frac{d}{d\tau}I = \Big(\alpha_0\langle e^{-a}\rangle^S + (1-\alpha_0)e^{-\langle a\rangle^{NS}}\Big)\alpha I(1-I) - \beta I,\\
&\frac{d}{d\tau}W = W\Big(\rho I + \sigma(a^*_\delta - \alpha_0\langle a\rangle^S - (1-\alpha_0)\langle a\rangle^{NS}) + \eta(a^*_\delta - W)\Big),\\
&\frac{d}{d\tau}\langle a\rangle^{NS} = \langle q\rangle(\gamma+\kappa\alpha_0)\Big(\frac{\gamma W(\tau) + \kappa\alpha_0\langle a\rangle^S}{\gamma+\kappa\alpha_0} - \langle a\rangle^{NS}\Big).
\end{aligned}
\tag{3.16}
$$

3.4. **Impact of stubborn individuals.** To qualitatively assess the impact of stubborn individuals, we suppose that the stubborn agents have all awareness 0, in particular $\langle a\rangle^S = 0$ and $\langle e^{-a}\rangle^S = 1$. Then system (3.16) is reduced to

$$
\begin{aligned}
&\frac{d}{d\tau}I = \Big(\alpha_0 + (1-\alpha_0)e^{-\langle a\rangle^{NS}}\Big)\alpha I(1-I) - \beta I,\\
&\frac{d}{d\tau}\langle a\rangle^{NS} = \langle q\rangle(\gamma+\kappa\alpha_0)\Big(\frac{\gamma W(\tau)}{\gamma+\kappa\alpha_0} - \langle a\rangle^{NS}\Big),\\
&\frac{d}{d\tau}W = W\Big(\rho I + \sigma(a^*_\delta - (1-\alpha_0)\langle a\rangle^{NS}) + \eta(a^*_\delta - W)\Big).
\end{aligned}
\tag{3.17}
$$

Let us first examine two extreme cases where $\gamma \gg 1$ or $\kappa \geq 1$ i.e. where the social dynamic evolves on a time scale much faster than those of the disease and of the government.

First if $\kappa \gg 1$ with $\kappa \gg \gamma$, then, intuitively, harmless social interactions, such as calls or texts, occur on a much faster time-scale than the infectious interactions. Also, they are more frequent than the interactions with the government, which is relatively absent from the public debate.

In that case $\langle a\rangle^{NS} \simeq 0$ so that

$$\frac{d}{d\tau}I \simeq \alpha I(1-I) - \beta I.$$

We thus recover the standard SIS dynamic and the disease will become endemic.

On the other hand, if $\gamma \gg 1$ with $\gamma \gg \kappa$, then the government is very active in the public debate and interactions between individuals and the government are much more frequent than between individuals. In that case $\langle a\rangle^{NS} \simeq W$ and

$$
\begin{aligned}
&\frac{d}{d\tau}I = \Big(\alpha_0 + (1-\alpha_0)e^{-W}\Big)\alpha I(1-I) - \beta I,\\
&\frac{d}{d\tau}W = W\Big(\rho I + \sigma(a^*_\delta - (1-\alpha_0)W) + \eta(a^*_\delta - W)\Big).
\end{aligned}
\tag{3.18}
$$

We can see that the endemic state exists if $\alpha_0\alpha/\beta > 1$, mainly inside the stubborn population, even if $W \to \infty$.

We now turn to the general case. First, up to multiplying the parameters $\alpha, \beta, \rho, \sigma, \eta$ by $\langle q\rangle(\gamma+\kappa\alpha_0)$ and changing the time-scale, we can assume that $\langle q\rangle(\gamma+\kappa\alpha_0) = 1$. We also let

$\chi = \kappa/\gamma$. Thus

$$
\begin{aligned}
&\frac{d}{d\tau}I = \Big(\alpha_0 + (1-\alpha_0)e^{-\langle a\rangle^{NS}}\Big)\alpha I(1-I) - \beta I,\\
&\frac{d}{d\tau}\langle a\rangle^{NS} = \frac{W(\tau)}{1+\chi} - \langle a\rangle^{NS},\\
&\frac{d}{d\tau}W = W\Big(\rho I + \sigma(a^*_\delta - (1-\alpha_0)\langle a\rangle^{NS}) + \eta(a^*_\delta - W)\Big).
\end{aligned}
\tag{3.19}
$$

We can prove the following result:

Teorem 3.5. *The system 3.19 has a non endemic equlibrium* $I = 0$, $\langle a\rangle^{NS} = \frac{a^*_\delta(\sigma+\eta)}{\sigma(1-\alpha_0)+\eta(1+\chi)}$, $W = \langle a\rangle^{NS}(1+\chi)$. *The equilibrium is locally asymptotically stable if and only if*

$$\Big(\alpha_0 + (1-\alpha_0)e^{-\langle a\rangle^{NS}}\Big)\,\alpha - \beta < 0.$$

We omit the proof, which follows by computing the Jacobian of the linearized system at the equilibrium point, and by observing that $\Big(\alpha_0 + (1-\alpha_0)e^{-\langle a\rangle^{NS}}\Big)\,\alpha - \beta$ is the only eigenvalue which can change signs, since the other two eigenvalues are always negative.

4. Conclusion and future works

In this work we analyzed a SIS model coupled with a social dynamics. Individual can reduce the rate of contagion by taking protection measures, and they adjust their awareness levels by interacting with other agents or the government. The government send a message and its intensity depends on some ideal level of awareness, big enough to avoid an endemic state, and also in the proportion of infected people.

We have modeled it at three levels:

- Agent-based model, where agents interact following the microscopic rules of the SIS and the social dynamics.
- A mean field approach, by obtaining a system of ordinary differential equations for the awareness actualization, the government signal and the epidemic dynamics. We characterized the equilibria and its local stability.
- A kinetic approach, where a Boltzmann-like equation coupled with ordinary differential equations was derived, together with its approximation with a coupled PDE-ODE system.

The existence and uniqueness of solutions for the Boltzmann-like equation coupled with the ordinary differential equations was proved by using a technique due to Bressan [8] (see also [3]), which has independent interest. Moreover, the grazing limit enable us to prove the existence of solution for the PDE-ODE system.

In the mean field approach we characterized the equilibria and its local stability. The theoretical results agree with simulations for the agent-based model.

In future works we plan to analyze the influence of random factors. For instance, the individual levels can fluctuate at random, and the measures taken can fail with certain probability. Also, the signal can be perturbed by some noise, and starting from a random initial condition for W will generate a transport equation for a probability density $f(W)dW$, as we studied in

[25]. Let us remark that in this work a random differential equation with delay was considered, which makes sense in this context since the forecast of the epidemic has a delay, which is greater in SEIRS type model, where E stands for exposed, and can include the incubation period of a disease. In both random and delayed dynamics, the long time behavior of the system, its equilibria, and their stability, pose a challenge and require different techniques than those used here.

On the other hand, certain measures of the government like lock-downs or travel restrictions were quickly implemented in the last two years, so we can consider that some signals propagates in a faster time scale than the epidemic, generating a slow-fast dynamics, i.e.,

$$\begin{aligned} I'(t) &= \langle e^{-a}\rangle \alpha I(t)(1-I(t)) - \beta I(t), \\ \varepsilon a_i'(t) &= -\gamma(a_i(t) - W(t)) \qquad i = 1,\dots,N, \\ \varepsilon W'(t) &= W(t)\Big(\rho I(t) + \sigma(a^* + \delta - \langle a\rangle_t) + \eta(a^* + \delta - W(t))\Big). \end{aligned} \tag{4.1}$$

We believe that this problem is more interesting in SIR/SEIR models, since the final size of the epidemic depends on the immediate reduction of the effective reproduction number. Moreover, the agent-government interaction, acts as a decentralized control, in opposition to more classical control approaches as in [4, 14].

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Appendix A. Linear stability analysis: proof of Theorem 2.3.

Suppose that $\alpha > \beta$. We linearize the system

$$\begin{aligned} I'(t) &= \langle e^{-a}\rangle \alpha I(t)(1-I(t)) - \beta I(t), \\ a_i'(t) &= -\gamma(a_i(t) - W(t)), \qquad i = 1,\dots,N, \\ W'(t) &= W(t)\Big(\rho I(t) + \sigma(a^*_\delta - \langle a\rangle_t) + \eta(a^*_\delta - W(t))\Big) \end{aligned}$$

around the equilibria $(0,\dots,0)$, $(1-\frac{\beta}{\alpha}, 0,\dots,0)$ and $(0, a^*_\delta,\dots,a^*_\delta, a^*_\delta)$.

For $(0,...,0)$ we obtain the matrix

$$A = \begin{pmatrix} \alpha-\beta & 0 & \dots & \dots & 0 & 0 \\ 0 & -\gamma & 0 & \dots & 0 & \gamma \\ 0 & 0 & -\gamma & \dots & 0 & \gamma \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -\gamma & \gamma \\ 0 & 0 & 0 & \dots & 0 & (\sigma+\eta)a^*_\delta \end{pmatrix}.$$

Notice that $\alpha - \beta > 0$ is a positive eigenvalue so $(0,\dots,0)$ is unstable.

Then, by linearizing around $(1-\frac{\beta}{\alpha},0,\dots,0)$ we get

$$A=\begin{pmatrix} \beta-\alpha & -\frac{\beta}{N}(1-\frac{\beta}{\alpha}) & \dots & \dots & -\frac{\beta}{N}(1-\frac{\beta}{\alpha}) & 0\\ 0 & -\gamma & 0 & \dots & 0 & \gamma\\ 0 & 0 & -\gamma & \dots & 0 & \gamma\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & -\gamma & \gamma\\ 0 & 0 & 0 & \dots & 0 & \rho(1-\frac{\beta}{\alpha})+(\sigma+\eta)a_\delta^* \end{pmatrix}.$$

Notice that $\rho(1-\frac{\beta}{\alpha})+(\sigma+\eta)a_\delta^*>0$ is a positive eigenvalue so that $(1-\frac{\beta}{\alpha},0,\dots,0)$ is unstable.

Finally, by linearizing around the equilibrium $P=(0,a_\delta^*,\dots,a_\delta^*,a_\delta^*)$ yields the $(N+2)\times(N+2)$ matrix

$$A=\begin{pmatrix} \beta(e^{-\delta}-1) & 0 & \dots & \dots & 0 & 0\\ 0 & -\gamma & 0 & \dots & 0 & \gamma\\ 0 & 0 & -\gamma & \dots & 0 & \gamma\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & -\gamma & \gamma\\ \rho a_\delta^* & -\frac{\sigma a_\delta^*}{N} & -\frac{\sigma a_\delta^*}{N} & \dots & -\frac{\sigma a_\delta^*}{N} & -\eta a_\delta^*. \end{pmatrix}.$$

Looking at the first file of A we see that $\beta(e^{-\delta}-1)$ is an eigenvalue which is negative if $\delta>0$ and positive if $\delta<0$. In particular if $\delta<0$ then P is unstable. Then, recall that A and its transpose A^T have the same eigenvalues, and

$$A^T+\gamma I=\begin{pmatrix} \beta(e^{-\delta}-1)+\gamma & 0 & \dots & 0 & \rho a_\delta^*\\ 0 & 0 & \dots & 0 & -\frac{\sigma a_\delta^*}{N}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & \vdots & \dots & 0 & -\frac{\sigma a_\delta^*}{N}\\ 0 & \gamma & \dots & \gamma & \gamma-\eta a_\delta^*. \end{pmatrix}$$

has rank 3 and so its kernel has dimension $(N+2)-3=N-1$. Thus $-\gamma$ is an eigenvalue of A^T, and A, with multiplicity $N-1$. Let us denote λ and μ the two remaining eigenvalues of A^T. Evaluating the trace of A^T we obtain

$$\lambda+\mu-(N-1)\gamma+\beta(e^{-\delta}-1)=\sum_i A_{ii}=\beta(e^{-\delta}-1)-(N-1)\gamma-\eta a_\delta^*$$

which gives

$$\lambda+\mu=-\eta a_\delta^*. \tag{A.1}$$

Next evaluating the determinant of A^T we first have

$$\det(A)=\lambda\mu(-\gamma)^{N-1}\beta(e^{-\delta}-1). \tag{A.2}$$

On the other hand,

$$\det(A)=\beta(e^{-\delta}-1)\det(B)$$

where

$$B = \begin{pmatrix} -\gamma & 0 & \dots & 0 & -\frac{\sigma a_\delta^*}{N} \\ 0 & -\gamma & \dots & 0 & -\frac{\sigma a_\delta^*}{N} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -\gamma & -\frac{\sigma a_\delta^*}{N} \\ \gamma & \gamma & \dots & \gamma & -\eta a_\delta^*). \end{pmatrix} \in \mathbb{R}^{(N+1)\times(N+1)}.$$

Adding to the last row of B the sum of the first N rows gives

$$\det(B) = \det \begin{pmatrix} -\gamma & 0 & \dots & 0 & -\frac{\sigma a_\delta^*}{N} \\ 0 & -\gamma & \dots & 0 & -\frac{\sigma a_\delta^*}{N} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -\gamma & -\frac{\sigma a_\delta^*}{N} \\ 0 & 0 & \dots & 0 & -(\eta+\sigma) a_\delta^*. \end{pmatrix} = (-1)^{N+1}\gamma^N(\eta+\sigma)a_\delta^*.$$

In view of (A.2) we conclude that

$$\lambda\mu = \gamma(\eta+\sigma)a_\delta^*. \tag{A.3}$$

Thus the two last unknown eigenvalues λ, μ of A^T satisfy

$$\lambda + \mu = -\eta a_\delta^*, \qquad \lambda\mu = \gamma(\eta+\sigma)a_\delta^*.$$

If $\lambda, \mu \in \mathbb{C}\backslash\mathbb{R}$ then $\mu = \bar{\lambda}$ and so $2Re(\lambda) = -\eta a_\delta^*$. If $\lambda, \mu \in \mathbb{R}$ then they must have same sign because $\lambda\mu > 0$ and so are negative because $\lambda + \mu < 0$. In any case we obtain $Re(\lambda), Re(\mu) < -C$ with $C > 0$ independent of N.

Recalling that the N others eigenvalues of A^T are $\beta(e^{-\delta} - 1)$ (simple, negative if $\delta > 0$), and $-\gamma$ (with mutiplicity $N-1$), we conclude that there exists $C > 0$ independent of N such that the eigenvalues of A have real part less then $-C$.

Appendix B. Well-posedness for the ODEs, Boltzmann-like system: proof of Theorem 3.1.

We denote $\mathcal{M}_b(\mathbb{R})$ the space of Borel signed finite measure on $\mathbb{R}$, and $\mathcal{M}_{b,+}([0,R])$ the subset of non-negative measures. Recall that the Total Variation (TV) norm and the Bounded Lipschitz (BL) norm of $f \in \mathcal{M}_b(\mathbb{R})$ are defined by

$$\|f\|_{TV} = \sup_{\|\phi\|_\infty \le 1} \int \phi \, df, \qquad \|f\|_{BL} = \sup_{\|\phi\|_{W^{1,\infty}} \le 1} \int \phi \, df \tag{B.1}$$

where $\|\phi\|_{W^{1,\infty}} := \max\{\|\phi\|_\infty, Lip(\phi)\} \le 1$, being $Lip(\phi)$ the Lipschitz constant of ϕ. We refer to [13] for general properties of these norms.

We rewrite the system (3.6) as

$$\begin{aligned} \frac{d}{dt} f_t &= Q[x(t), f_t], \\ x'(t) &= F[f_t](x(t)), \end{aligned} \tag{B.2}$$

where $x(t) = (x_1(t), x_2(t)) := (W(t), I(t))$, $Q : \mathbb{R}^2 \times \mathcal{M}_b(\mathbb{R}) \to \mathcal{M}_b(\mathbb{R})$ is defined by

$$(Q[x,f],\phi) = \int \phi(a + \gamma(x_1 - a)) \, df(a) - \int \phi(a)] \, df(a) \qquad \phi \in C_b(\mathbb{R}),$$

and, given $f \in \mathcal{M}_b(\mathbb{R})$, the vector-field $F[f] : \mathbb{R}^2 \to \mathbb{R}^2$ has components

$$\begin{aligned} F_1[f](x) &= x_1(\rho x_2 - \eta x_1 - \sigma \langle a \rangle_g + (\sigma+\eta) a_\delta^*), \\ F_2[f](x) &= x_2 \Big(\alpha \langle e^{-a} \rangle_f (1 - x_2) - \beta \Big), \end{aligned}$$

with $\langle a \rangle_f := \int_{-\infty}^{+\infty} a \, df(a)$ and $\langle e^{-a} \rangle_f := \int_{-\infty}^{+\infty} e^{-a} \, df(a)$.

Given $f \in \mathcal{M}_b(\mathbb{R})$, denote $x[f](t)$ the solution of

$$\begin{aligned} &\frac{d}{dt} x[f](t) = F[f_t](x[f](t)) \\ &x[f](0) = x_0 \end{aligned} \tag{B.3}$$

From now on we fix an initial condition $x(0)$ such that $x_1(0) \geq 0$ and $x_2(0) \in [0,1]$ and then some constant $N > 0$ such that

$$N > \max \Big\{ x_1(0), \frac{\rho + (\sigma+\eta) a_\delta^*}{\eta} \Big\}. \tag{B.4}$$

Proposition B.1. *Assume that $f : [0,+\infty) \to \mathcal{M}_b(\mathbb{R})$ is continuous for the BL-norm with f_t non-negative and supported in some fixed interval $[0,R]$ for any $t \in [0,T]$. Then $x[f](t)$ exists and belongs to $[0,N] \times [0,1]$ for any $t \in \mathbb{R}$.*

Moreover for any $0 \leq s \leq t$,

$$|x_1[f](t) - x_1[f](s)| \leq N(\rho + (\sigma+\eta) a^*)(t-s). \tag{B.5}$$

Eventually consider another continuous function $\bar{f} : [0,+\infty) \to \mathcal{M}_{b,+}([0,R])$ and fix some $T > 0$. Assume that $\|f_t\|_{TV}, \|\bar{f}_t\|_{TV} \leq R$ for $t \in [0,T]$ (up to increasing the support). Then for any t,

$$|x[f](t) - x[\bar{f}](t)| \leq N(R\sigma + \alpha e^R) t e^{M_R t} \max_{0 \leq s \leq t} \|f_s - \bar{f}_s\|_{BL} \tag{B.6}$$

where M_R is defined in (B.8).

Proof. Since the functions $a \mapsto a$ and $a \mapsto e^{-a}$ are bounded Lipschitz in $[0,R]$, the maps $t \to \langle a \rangle_{f_t}$ and $t \to \langle e^{-a} \rangle_{f_t}$ are continuous. Thus the solution $x(t) := x[f](t)$ of (B.3) exists on some interval around 0.

Since $F_1[g](0,x_2) = F_2[g](x_1,0) = F_2[g](x_1,1) = 0$ for any $g \in \mathcal{M}_b(\mathbb{R})$ and $x \in \mathbb{R}^2$, and $x_1(0) \geq 0$, $x_2(0) \in [0,1]$, we have $x_1(t) \geq 0$ and $x_2(t) \in [0,1]$. Since f is non-negative and supported in $[0,+\infty)$, we have $\langle a \rangle_f \geq 0$ so that

$$x_1'(t) \leq \gamma x_1(t) \Big(\frac{\rho + (\sigma+\eta) a_\delta^*}{\eta} - x_1 \Big).$$

Recalling the definition of N, it follows that $x_1(t) \leq N$ for any t. Thus $x(t)$ stays in the compact $[0,N] \times [0,1]$ and so exists for any $t \in \mathbb{R}$.

To prove (B.5), notice that for any $\tau \geq 0$, $|F_1[f_\tau](x[f](\tau))| \leq N(\rho + (\sigma+\eta) a_\delta^*)$. Then writing

$$x_1[f](t) - x_1[\bar{f}](s) = \int_s^t F_1[f_\tau](x[f](\tau))$$

yields the result.

We eventually prove (B.6). For ease of notation we let $x(t) := x[f](t)$ and $\bar{x}(t) := x[\bar{f}](t)$. Then

$$x(t) - \bar{x}(t) = \int_0^t F[f_s](x(s)) - F[\bar{f}_s](\bar{x}(s))\, ds.$$

Using Proposition B.2 below (with $A = N$) we obtain for $t \in [0, T]$ that

$$\begin{aligned} |x(t) - \bar{x}(t)| &\leq \int_0^t M_R|x(s) - \bar{x}(s)| + N(R\sigma + \alpha e^R)\|f_s - \bar{f}_s\|_{BL}\, ds \\ &\leq N(R\sigma + \alpha e^R)t \max_{0\leq s\leq t} \|f_s - \bar{f}_s\|_{BL} + M_R \int_0^t |x(s) - \bar{x}(s)|\, ds. \end{aligned}$$

We deduce (B.6) by applying Gronwall inequality.

□

In the proof we used the following result:

Proposition B.2. *For any $A > 0$, any $x, \bar{x} \in \bar{B}(0, A)$ and any $f, \bar{f} \in \mathcal{M}_b([-R, R])$ such that $\|f\|_{TV}, \|\bar{f}\|_{TV} \leq R$, we have*

$$|F[f](x) - F[\bar{f}](\bar{x})| \leq M_R|x - \bar{x}| + A(R\sigma + \alpha e^R)\|f - \bar{f}\|_{BL} \tag{B.7}$$

($\mathbb{R}^2$ is endowed with the norm $\|.\|_1$) with

$$M_R := \sup_g \|\nabla F[g]\|_{L^\infty(\bar{B}(0,A))} \tag{B.8}$$

where the supremum is taken over all $g \in \mathcal{M}_b([-R, R])$ such that $\|g\|_{TV} \leq R$.

Notice that for any such g, $|\langle a\rangle_g| \leq R^2$ and $|\langle e^{-a}\rangle_g| \leq Re^R$.

Proof. We write

$$|F[f](x) - F[\bar{f}](\bar{x})| \leq |F[f](x) - F[\bar{f}](x)| + |F[\bar{f}](x) - F[\bar{f}](\bar{x})|.$$

The second term in the right hand side is less than $|\nabla F[\bar{f}]\|_{L^\infty(\bar{B}(0,A))}|x - \bar{x}| \leq M_R|x - \bar{x}|$. Concerning the first term notice that the maps $a \mapsto a$ and $a \mapsto e^{-a}$ are bounded Lipschitz in $[-R, R]$. Thus

$$\begin{aligned} |F_1[f](x) - F_1[\bar{f}](x)| &= |x_1|\sigma\Big|\int_{-R}^R a\, d(\bar{f} - f)\Big| \\ &\leq A\sigma R\|\bar{f} - f\|_{BL}, \end{aligned}$$

and

$$\begin{aligned} |F_2[f](x) - F_2[\bar{f}](x)| &= |x_2|\alpha\Big|\int_{-R}^R e^{-a}\, d(\bar{f} - f)\Big| \\ &\leq A\alpha e^R\|\bar{f} - f\|_{BL}. \end{aligned}$$

We deduce (B.7). □

We can now rewrite (B.2) as

$$\frac{d}{dt} f_t = Q[x[f](t), f_t] =: \bar{Q}[f_t]. \tag{B.9}$$

We will prove the existence of a solution to this equation using Bressan's idea [8] as exposed in [3].

We first recall the existence and uniqueness result proved in [3][Theorem 6.1] which deals with the equation

$$\partial_t f = \tilde{Q}[f] \qquad \text{in } [0,T) \times E \tag{B.10}$$

$$f(0) = f_0 \in S \tag{B.11}$$

where E is a Banach space, S is a closed bounded convex subset of E, and

$$\tilde{Q} : C([0,T],S) \to C([0,T],E)$$

is a causal operator in the sense that $\tilde{Q}[f](t) = \tilde{Q}[f 1_{[0,t]}](t)$ for any $f \in C([0,T],E)$.

We assume the following conditions on Q:

- Hölder continuity: for any $f,g \in C([0,T],S)$ and any times $0 \le s \le t \le T$, there exists $\beta \in (0,1)$ such that

$$\|\tilde{Q}[f](t) - \tilde{Q}[g](s)\| \le C\Big(\max_{0\le\tau\le s} \|f(\tau)-g(\tau)\|^\beta + \|f(t)-g(s)\|^\beta + |t-s|^\beta \Big) \tag{B.12}$$

- Sub-tangent condition: for any $f \in C([0,T],S)$,

$$\liminf_{h\to 0^+} \frac{1}{h} \sup_{0\le t\le T} \{dist(f(t)+h\tilde{Q}[f](t),S)\} = 0 \tag{B.13}$$

- One-sided Lipschitz condition: for any $f,g \in C([0,T],S)$ and any $t \in [0,T]$,

$$\int_0^t \Big[f(s)-g(s), \tilde{Q}[f](s) - \tilde{Q}[g](s)\Big]\,ds \le L \int_0^t \|f(s)-g(s)\|\,ds \tag{B.14}$$

  where $[\Phi,\phi] := \lim_{h\to 0^-} \frac{1}{h}[\|\Phi + h\Phi\| - \|\Phi\|]$.

Under these assumptions it is proved in [3][Theorem 6.1] that Eq. (B.10) has a unique solution in $C([0,T),S)\cap C^1((0,T),E)$.

Using this result we can prove the existence of a unique solution to (B.9). Let $R = \max\{R_0, N\}$ where R_0 is such that supp $f_0 \subset [-R_0,R_0]$. We take

$$E = \{f \in \mathcal{M}_b([0,+\infty]) : \text{supp} f \subset [0,R],\ \|f\|_{TV} \le 2\},$$

and

$$S = \{f \in E,\ f \text{ is a probability measure}\}.$$

We endow S with the Bounded Lipschitz (BL) norm defined in (B.1). When endowed with the BL norm, E is complete and S is a closed convex subset of E.

The operator

$$\tilde{Q}[f]_t := Q[x[f](t), f_t]$$

is causal and satisfies the following properties:

Proposition B.3. *The operator $\tilde{Q}$ satisfies the following properties:*

(i) *For any $f,g \in C([0,T],S)$ and any $0 \le s \le t$,*

$$|\tilde{Q}[f](t) - \tilde{Q}[g](s)| \le C\Big(\max_{0\le\tau\le s} \|f_\tau - g_\tau\|_{BL} + \|f_t - f_s\|_{BL} + |t-s|\Big)$$

where C depends only on N, R and the modelling parameters $\rho, \sigma, \gamma, a^*$.

(ii) *For any $f \in C([0,T],S)$, any $t \in [0,T]$ and any $h \in [0,1]$, $f_t + h\tilde{Q}[f]_t \in S$.*

This Proposition will be a consequence of some simple properties of $Q[x,f]$ for $x \in \mathbb{R}^2$, $f \in M_b(\mathbb{R})$. It will be convenient to write $Q[x,f]$ as

$$Q[x,f] = Q^+[x,f] - f$$

where $Q^+[x,f] \in \mathcal{M}_b(\mathbb{R})$ is defined as

$$(Q^+[x,f],\phi) = \int \phi((1-\gamma)a + \gamma x_1)\,df(a) \qquad \phi \in C_b(\mathbb{R}). \tag{B.15}$$

Proposition B.4. *The following properties hold:*

(i) *if $f \geq 0$ then $Q^+[x,f]$ is a non-negative measure.*
(ii) *if $supp(f) \subset [0,\hat{R}]$ and $x_1 \in [0,\tilde{R}]$ then $Q^+[x,f]$ is supported in $[0,\bar{R}]$ with $\bar{R} = \max\{\hat{R},\tilde{R}\}$.*
(iii) *For any $x,\bar{x} \in \mathbb{R}^2$ and any $f,\bar{f} \in P(\mathbb{R})$, it holds that*

$$\|Q[x,f] - Q[\bar{x},\bar{f}]\|_{BL} \leq \gamma|x_1 - \bar{x}_1| + 2\|f - \bar{f}\|_{BL}. \tag{B.16}$$

Proof. If $f \geq 0$ then for any non-negative ϕ, $(Q^+[x,f],\phi) \geq 0$ in view of (B.15). We deduce (i).

Concerning (ii), just remark that for any $a \in \text{supp}(f)$ and any $\gamma \in [0,1]$, we have $(1-\gamma)a + \gamma x_1 \in [0,\bar{R}]$. Thus $(Q^+[x,f],\phi) = 0$ if ϕ has support in $\mathbb{R}\backslash[0,\bar{R}]$. This proves (ii).

We eventually prove (B.16). For any $\phi \in W^{1,\infty}(\mathbb{R})$ with norm ≤ 1 we have

$$(Q[x,f] - Q[\bar{x},\bar{f}],\phi) = (Q^+[x,f] - Q^+[\bar{x},\bar{f}],\phi) + (\bar{f} - f,\phi)$$

with $|(\bar{f} - f,\phi)| \leq \|\bar{f} - f\|_{BL}$ and

$$\begin{aligned}(Q^+[x,f] - Q^+[\bar{x},\bar{f}],\phi) &= \int \phi((1-\gamma)a + \gamma x_1) - \phi((1-\gamma)a + \gamma\bar{x}_1)\,df(a) \\ &\quad + \int \phi((1-\gamma)a + \gamma\bar{x}_1)\,d(f - \bar{f})(a) \\ &\leq \gamma|x_1 - \bar{x}_1| + \|\bar{f} - f\|_{BL}\end{aligned}$$

In the last inequality, we used that ϕ is 1-Lipschitz to bound the first integral, and we bound the second integral by $\|\bar{f} - f\|_{BL}$ times the $W^{1,\infty}$-norm of the function $a \to \phi((1-\gamma)a + \gamma\bar{x}_1)$ which is less than 1. □

We can now prove Proposition B.3.

Proof of Proposition B.3. We prove (i) by writing

$$\begin{aligned}|\tilde{Q}[f](t) - \tilde{Q}[g](s)| &\leq |\tilde{Q}[f](t) - \tilde{Q}[f](s)| + |\tilde{Q}[f](s) - \tilde{Q}[g](s)| \\ &= |Q[x[f](t),f_t] - Q[x[f](s),f_s]| + |Q[x[f](s),f_s] - Q[x[g](s),g_s]| \\ &=: I + II.\end{aligned}$$

In view of (B.16), (B.5) and (B.6), we can bound I and II as follow:

$$\begin{aligned}I &\leq \gamma|x_1[f](t) - x_1[f](s)| + 2\|f_t - f_s\|_{BL} \\ &\leq \gamma N(\rho + (\sigma+\eta)a^*_\delta)(t-s) + 2\|f_t - f_s\|_{BL}.\end{aligned}$$

and

$$\begin{aligned} II &\le \gamma|x_1[f](s) - x_1[g](s)| + 2\|f_s - g_s\|_{BL} \\ &\le \gamma N R\sigma s e^{M_R s} \max_{0\le\tau\le s} \|f_\tau - g_\tau\|_{BL} + 2\|f_t - f_s\|_{BL}. \end{aligned}$$

We prove (ii). Using (B.15) we can write $f_t + h\tilde{Q}[f]_t = (1-h)f_t + h\tilde{Q}^+[f]_t$ with $\tilde{Q}^+[f]_t = Q^+[x[f](t), f_t]$. Since f_t is a probabilty measure, $\tilde{Q}^+[f]_t$ is a non-negative measure and so is $(1-h)f_t + h\tilde{Q}^+[f]_t$ for $h \in [0,1]$. Moreover

$$(f_t + h\tilde{Q}[f]_t, 1) = (1-h)(f_t, 1) + h(Q^+[x[f](t), f_t], 1) = (f_t, 1) = 1.$$

Thus $f_t + h\tilde{Q}[f]_t$ is a probability measure. In particular its TV norm is 1.

It remains to prove that $f_t + h\tilde{Q}[f]_t$ is supported in $[0,R]$. Since $f_t \in E$, it is supported in $[0,R]$. We show that $Q^+[x[f](t), f_t]$ is supported in $[0,R]$ as well. Recall that $x_1[f](t) \in [0,N]$ by Proposition B.1. According to what we said just before beginning the proof, $Q^+[x[f](t), f_t]$ is supported in $[0,\bar{R}]$ with $\bar{R} = \max\{N,R\}$ i.e. $\bar{R} = R$ by the definition of R.

□

We can now conclude the proof of the existence of a unique solution to (B.9) in $[0,T]$. According to Proposition B.3, the operator $\tilde{Q}$ satisfies the three hypothesis in [3][Theorem 6.1]. More precisely, (i) and (ii) of Proposition B.3 gives respectively the Holder condition (B.12) with $\beta = 1$ and the sub-tangent condition (B.13). Moreover it is easy to see that the one-sided Lipschitz condition (B.14) automatically holds when the Holder condition holds with $\beta = 1$. Since T is arbitrary, we obtain existence and uniqueness of a global solution.

Appendix C. Proof of Theorem 3.2.

We rewrite the system (3.7) as

$$\begin{aligned} &\partial_t g_t + \partial_a(v[x(t)](a) g_t) = 0, \\ &x'(t) = F[g_t](x(t)), \end{aligned} \tag{C.1}$$

where

$$\begin{aligned} &x(t) = (x_1(t), x_2(t)) := (W(t), I(t)), \\ &v[x(t)](a) = x_1(t) - a, \end{aligned}$$

and, given $g \in P(\mathbb{R})$, the vector-field $F[g] : \mathbb{R}^2 \to \mathbb{R}^2$ has components

$$\begin{aligned} F_1[g](x) &= x_1(\rho x_2 - \eta x_1 - \sigma\langle a\rangle_g + (\sigma+\eta)a^*_\delta), \\ F_2[g](x) &= x_2\Big(\alpha\langle e^{-a}\rangle_g(1-x_2) - \beta\Big), \end{aligned}$$

with $\langle a\rangle_g := \int_{-\infty}^{+\infty} a\, dg(a)$ and $\langle e^{-a}\rangle_g := \int_{-\infty}^{+\infty} e^{-a}\, dg(a)$.

We fix initial conditions $x_0 = (W(0), I(0)) \in \mathbb{R}^2$ and $g_0 \in P(\mathbb{R})$ with compact support. Let $R_0 > 0$ be such that $\mathrm{supp}(g_0) \subset [-R_0, R_0]$.

We want to rewrite (C.1) as a fixed point problem. Given some $T > 0$, to be chosen later, consider the space

$$X_T = C([0,T], P(\mathbb{R})) \times C([0,T], \mathbb{R}^2)$$

endowed with the sup-norm. Since $P(\mathbb{R})$ is complete, X_T is complete. Then consider the set Y_T consisting of the pairs $(g,x) \in X_T$ such that

$$\begin{aligned} &g(0) = g_0, \qquad x(0) = x_0, \\ &|x(t) - x_0| \leq 1 \qquad t \in [0,T], \\ &\mathrm{supp}(g_t) \subset [-2R_0, 2R_0] \qquad t \in [0,T]. \end{aligned}$$

Then Y_T is complete as a closed subset of X_T.

Given $(g,x) \in Y_T$, define $\Gamma(g,x) : [0,T] \to M_b(\mathbb{R}) \times \mathbb{R}^2$ by

$$\Gamma(g,x)_t = (\Gamma^1(g,x)_t, \Gamma^2(g,x)_t) := \Big(T_t^{v[x_t]} \sharp g_0, x_0 + \int_0^t F[g_s](x(s))\, ds \Big)$$

where $T_t^{v[x_t]}$ is the flow of $v[x_t]$ i.e. $T_t^{v[x_t]} = T_{0,t}^{v[x_t]}$ with, for any $a \geq 0$ and any $t \geq s \geq 0$,

$$\frac{d}{dt} T_{s,t}^{v[x_t]}(a) = v[x_t](T_{s,t}^{v[x_t]}(a)),\ t > s, \qquad T_{s,s}^{v[x_t]}(a) = a.$$

Then for a small enough T, (g,x) solves (C.1) in $[0,T]$ with initial conditions (g_0, x_0) if and only if $(g,x) \in Y_T$ and (g,x) is a fixed-point of Γ.

We prove the existence of such a fixed point by applying the classical Banach fixed-point theorem to Γ in Y_T. We thus need to show that (i) $\Gamma(Y_T) \subset Y_T$ and also that (ii) Γ is a strict contraction.

Fix some $(g,x) \in Y_T$. Since for $t \in [0,T]$, $|x(t)| \leq 1 + |x_0|$ and g_t is supported in $[-2R_0, 2R_0]$, we have $\max_{0 \leq s \leq T} |F[g_s](x(s))| \leq C$ with $C = C(|x_0|, R_0)$, and then $\Gamma^2(g,x)_t$ is continuous in $t \in [0,T]$. Moreover since $v[x_t]$ is bounded and globally Lipschitz in $[0,T]$, the map $t \to T_{s,t}^{v[x_t]}(a)$ is well-defined and continuous in $[0,T]$ for any $a \in \mathbb{R}$. Thus for any $\phi \in C_b(\mathbb{R})$, $(\Gamma^1(g,x)_t, \phi) = \int_{\mathbb{R}} \phi(T_t^{v[x_t]}(a))\, dg_0(a)$ is continuous in t by the Dominated Convergence Theorem. So far we proved that $\Gamma(g,x) \in X_T$. Moreover $\Gamma(g,x)_{t=0} = (g_0, x_0)$ obviously. Also $|\Gamma^1(g,x)_t - x_0| \leq \int_0^T F[g_s](x(s))\, ds \leq CT$ which is less than 1 choosing T small enough depending on $|x_0|$ and R_0. Eventually since for any $z \in \mathbb{R}$ and any $s \in [0,T]$,

$$|v[x(s)](z)| \leq |x_1(s)| + |z| \leq 1 + |x_0| + |z|,$$

we have

$$|T_t^{v[x_t]}(a)| \leq |a| + \int_0^t |v[x_s](T_s^{v[x_s]}(a))|\, ds \leq |a| + T(1 + |x_0|) + \int_0^t |T_s^{v[x_s]}(a)|\, ds.$$

Using Gronwall inequality we obtain

$$|T_t^{v[x_t]}(a)| \leq [|a| + T(1 + |x_0|)] e^T.$$

If $a \in [-R_0, R_0]$ we thus have $|T_t^{v[x_t]}(a)| \leq 2R_0$ choosing T small enough depending only on $|x_0|$ and R_0. Now for any smooth function ϕ with compact suuport in $\{|a| > 2R_0\}$ we deduce

$$(\Gamma^1(g,x)_t, \phi) = \int_{-R_0}^{R_0} \phi(T_t^{v[x_t]}(a))\, dg_0(a) = 0.$$

We conclude that $\Gamma^1(g,x)_t$ is supported in $[-2R_0, 2R_0]$ for any $t \in [0,T]$. We thus proved that we can choose T small enough depending only on $|x_0|$ and R_0 such that $\Gamma(Y_T) \subset Y_T$.

We now verify that Γ is a strict contraction for T small. First notice that there exist constants L depending only on R_0 and $|x_0|$ such that for any $g, \bar{g} \in P(\mathbb{R})$ supported in $[-R_0, R_0]$ and any $x, \bar{x} \in B(x_0, 1)$, there hold

$$|F[g](x) - F[g](\bar{x})| \leq L_1|x - \bar{x}|,$$
$$|F[g](\bar{x}) - F[\bar{g}](\bar{x})| \leq L_2 W_1(g, \bar{g}).$$

The first inequality is clear. For the second one notice that

$$|F_1[g](\bar{x}) - F_1[\bar{g}](\bar{x})| = \sigma|\bar{x}_1|\Big|\int_{-2R_0}^{2R_0} a\, d(g - \bar{g})\Big| \leq \sigma(1 + |x_0|)W_1(g, \bar{g}).$$

The same kind of inequality holds for F_2. As a consequence of these two inequalities we obtain

$$|F[g](x) - F[\bar{g}](\bar{x})| \leq L_3\Big(|x - \bar{x}| + W_1(g, \bar{g})\Big)$$

with $L_3 = \max\{L_1, L_2\}$. Given $(g, x), (\bar{g}, \bar{x}) \in Y_T$, we then deduce from this inequality that for any $t \in [0, T]$,

$$\begin{aligned} |\Gamma^2(g, x)_t - \Gamma^2(\bar{g}, \bar{x})_t| &\leq L_3 \int_0^t |x(s) - \bar{x}(s)| + W_1(g_s, \bar{g}_s)\, ds \\ &\leq L_3 T(\|x - \bar{x}\|_\infty + \max_{0\leq t\leq T} W_1(g_t, \bar{g}_t)) \end{aligned} \tag{C.2}$$

Moreover denoting $T_t := T_t^{v[x_t]}$ and $\bar{T}_t := T_t^{v[\bar{x}_t]}$ we have

$$\begin{aligned} |T_t(a) - \bar{T}_t(a)| &\leq \int_0^t |v[x(s)](T_t(a)) - v[\bar{x}(s)](\bar{T}_t(a))|\, ds \\ &\leq \int_0^t |x_1(s) - \bar{x}_1(s)| + |T_t(a) - \bar{T}_t(a)|\, ds \end{aligned}$$

so that by Gronwall inequality

$$|T_t(a) - \bar{T}_t(a)| \leq T\|x - \bar{x}\|_\infty e^t.$$

Then for any 1-Lipschitz function ϕ and any $a \in \mathbb{R}$, we obtain

$$|\phi(T_t(a)) - \phi(\bar{T}_t(a))| \leq |T_t(a) - \bar{T}_t(a)| \leq T\|x - \bar{x}\|_\infty e^t.$$

and then

$$(\Gamma^1(g, x)_t - \Gamma^1(\bar{g}, \bar{x})_t, \phi) \leq \int_{-\infty}^{+\infty} \phi(T_t(a)) - \phi(\bar{T}_t(a))\, dg_0(a) \leq T\|x - \bar{x}\|_\infty e^t$$

Taking the supremum over all such ϕ we obtain

$$\max_{0\leq t\leq T} W_1(\Gamma^1(g, x)_t, \Gamma^1(\bar{g}, \bar{x})_t) \leq Te^T\|x - \bar{x}\|_\infty.$$

With (C.2) we eventually obtain

$$\begin{aligned} \|\Gamma(g, x) - \Gamma(\bar{g}, \bar{x})\|_{X_T} &= \max_{0\leq t\leq T} W_1(\Gamma^1(g, x)_t, \Gamma^1(\bar{g}, \bar{x})_t) + \max_{0\leq t\leq T} |\Gamma^1(g, x)_t - \Gamma^1(\bar{g}, \bar{x})_t| \\ &\leq T(e^T + L_3)\|x - \bar{x}\|_\infty + L_3 T \max_{0\leq t\leq T} W_1(g_t, \bar{g}_t)) \end{aligned}$$

We can thus choose T small enough depending only on L_3, and so only on $|x_0|$ and R_0, such that Γ is a strict contraction.

Applying the Banach fixed point Theorem we obtain the existence of a unique fixed point of Γ in Y_T and thus of a unique solution to (C.1) in $[0,T]$. Iterating this argument we obtain a unique maximal solution (g,x) defined on a maximal interval time $[0,T^*)$ with $T^* \le \infty$.

Now let us assume that g_0 is supported in $[0,+\infty)$ and that $x(0)=(W(0),I(0)) \in [0,+\infty)\times[0,1]$. We will prove that $T^*=+\infty$, g_t is supported in a fixed compact of $[0,+\infty)$, x_1 is non-negative and bounded, and $x_2(t)\in[0,1]$, thus concluding the proof of Theorem 3.2.

Since $F_1[g](0,x_2)=F_2[g](x_1,0)=F_2[g](x_1,1)=0$ for any g,x_1,x_2, and $x_1(0)\ge 0$, $x_2(0)\in[0,1]$, we have $x_1(t)\ge 0$ and $x_2(t)\in[0,1]$. Using that $x_1(t)\ge 0$, we have for any $a\ge 0$ that $T_t(a)\ge 0$. Indeed the r.h.s. of $\frac{d}{dt}T_t(a)=x_1(t)-T_t(a)$ is non-negative when $T_t(a)=0$. Thus if g_0 is supported in $[0,+\infty)$ we deduce that $g_t=T_t\sharp g_0$ is also supported in $[0,+\infty)$. We can now bound x_1. Since $x_2\le 1$ and $\langle a\rangle_{g_t}\ge 0$, we can write

$$x_1'(t) \le \gamma x_1(t)\Big(\frac{\rho+(\sigma+\eta)a^*_\delta}{\eta}-x_1\Big).$$

Taking some $M>\max\{x_1(0),\frac{\rho+(\sigma+\gamma)a^*}{\gamma}\}$, it follows that $x_1(t)\le M$. As a consequence g_t is supported in a fixed compact for all t. Indeed Let R_0 be such that $\mathrm{supp}(g_0)\subset[0,R_0]$ with $R_0>M$. Then for any $a\in[0,R_0]$, $T_t(a)\le R_0$ for any t since the r.h.s. of $\frac{d}{dt}T_t(a)=x_1(t)-T_t(a)$ is less than $M-T_t(a)$ which is ≤ 0 when $T_t(a)=R_0$. Thus $g_t=T_t\sharp g_0$ is supported in $[0,R_0]$ for any t. Eventually since $g_t=T_t\sharp g_0$ is supported in $[0,R_0]$ and $x(t)\in[0,M]\times[0,1]$, we have $T^*=+\infty$ i.e. the solution (g,x) is defined for all $t\ge 0$.

Appendix D. Proof of Theorem 3.3.

We fix an initial condition (f_0,W_0,I_0) such that $f_0\in P([0,+\infty)$ has compact support, $W_0\ge 0$ and $I_0\in[0,1]$. Denote $(f_t^\gamma,W^\gamma(t),I^\gamma(t))$ the solution of

$$\begin{aligned}
&\frac{d}{dt}\int \phi(a)\,df_t(a)=\int[\phi(a')-\phi(a)]\,df_t^\gamma(a) \qquad \phi\in C_b([0,+\infty)),\\
&\frac{1}{\gamma}\frac{d}{dt}W^\gamma(t)=W(t)\Big(\rho I^\gamma(t)+\sigma(a^*_\delta-\langle a\rangle)_t+\eta(a^*_\delta-W^\gamma(t))\Big),\\
&\frac{1}{\gamma}\frac{d}{dt}I^\gamma(t)=\alpha\langle e^{-a}\rangle I^\gamma(t)(1-I^\gamma(t))-\beta I^\gamma(t).
\end{aligned} \tag{D.1}$$

where $a'=a+\gamma(W^\gamma(t)-a)$, $\langle a\rangle=\int a\,df_t^\gamma$ and $\langle e^{-a}\rangle=\int e^{-a}\,df_t^\gamma$. Since

$$\frac{1}{\gamma}\frac{d}{dt}W^\gamma(t) \quad\le\quad \eta W^\gamma(t)\Big(\frac{\rho+(\sigma+\eta)}{\eta}a^*_\delta-W^\gamma(t))\Big)$$

we have $0\le W^\gamma(t)\le\max\{W_0,\frac{\rho+(\sigma+\eta)}{\eta}\}$. We can thus take $R\ge 0$ such that f_0 is supported in $[0,R]$ and $W^\gamma(t)\in[0,R]$. As in the proof of Theorem 3.2 it follows that f_t^γ is supported in $[0,R]$ for any $t\ge 0$ and any γ.

Given $\phi\in C^2([0,R])$, we approximate $\phi(a')-\phi(a)$ using Taylor formula by

$$\begin{aligned}
\phi(a')-\phi(a)&=\phi'(a)(a'-a)+\frac{1}{2}\phi''(\xi)(a-a')^2\\
&=\phi'(a)\gamma(W^\gamma(t)-a)+\frac{1}{2}\phi''(\xi)\gamma^2(W^\gamma(t)-a)^2,
\end{aligned}$$

where $\xi = \theta a + (1-\theta)a'$ for some $\theta \in (0,1)$. We thus obtain

$$\frac{1}{\gamma}\frac{d}{dt}\int \phi(a)\, df_t^\gamma(a) = \int \phi'(a)(W^\gamma(t)-a)\, df_t^\gamma(a) + R^\gamma(t), \tag{D.2}$$

where

$$R^\gamma(t) = \frac{\gamma}{2}\int \phi''(\xi)(W^\gamma(t)-a)^2\, df_t^\gamma(a).$$

We bound $R^\gamma(t)$ by

$$|R^\gamma(t)| \le \gamma\|\phi''\|_\infty \int W^\gamma(t)^2 + a^2\, df_t^\gamma(a) \le 2\gamma\|\phi''\|_\infty R^2,$$

where we used that $W^\gamma(t) \in [0,R]$ and f_t^γ is supported in $[0,R]$ for any $t \ge 0$ and any γ. Letting $\tau = \gamma t$ and $g_\tau^\gamma := f_t^\gamma$, $\tilde W^\gamma(\tau) = W^\gamma(t)$ and $\tilde I^\gamma(\tau) = I^\gamma(t)$, we thus obtain

$$\frac{d}{d\tau}\int \phi(a)\, dg_\tau^\gamma(a) = \int \phi'(a)(\tilde W^\gamma(t)-a)\, dg_\tau^\gamma(a) + R^\gamma(\tau/\gamma).$$

Integrating between s and t with $s<t$ gives

$$\int \phi\, d(g_t^\gamma - g_s^\gamma) \le \int_s^t \int \phi'(a)(\tilde W^\gamma(t)-a)\, dg_\tau^\gamma(a) d\tau + 2\gamma\|\phi''\|_\infty R^2(t-s).$$

We denote $X = C^2([0,R])$ with the usual norm $\|\phi\|_X = \|\phi\|_\infty + \|\phi'\|_\infty + \|\phi''\|_\infty$, and define the norm $\|\mu\|$ of $\mu \in P([0,R])$ by

$$\|\mu\| := \sup_{\phi\in X, \|\phi\|_X \le 1} \int \phi\, d\mu. \tag{D.3}$$

Thus for any $\gamma > 0$ and any $s,t \ge 0$,

$$\|g_t^\gamma - g_s^\gamma\| \le C|t-s| \qquad C = 2R(1+2\gamma R). \tag{D.4}$$

Independently,

$$\begin{aligned} &\frac{d}{d\tau}\tilde W^\gamma(\tau) = \tilde W^\gamma(\tau)\Big(\rho \tilde I^\gamma(\tau) + \sigma(a_\delta^* - \langle a\rangle) + \eta(a_\delta^* - \tilde W^\gamma(\tau))\Big),\\ &\frac{d}{d\tau}\tilde I^\gamma(t) = \alpha\langle e^{-a}\rangle \tilde I^\gamma(\tau)(1-\tilde I^\gamma(\tau)) - \beta \tilde I^\gamma(\tau), \end{aligned} \tag{D.5}$$

with the slight abuse of notation $\langle a\rangle = \int a\, dg_\tau^\gamma$ and $\langle e^{-a}\rangle = \int e^{-a}\, dg_\tau^\gamma$. Since $\tilde W^\gamma(\tau) \in [0,R]$ and $\tilde I^\gamma(\tau) \in [0,1]$ for any γ and any $\tau \ge 0$, we have

$$\Big|\frac{d}{d\tau}\tilde W^\gamma(\tau)\Big| \le \tilde W^\gamma(\tau)(\rho+\sigma+\eta) \le R(\rho+\sigma+\eta)$$

and

$$\Big|\frac{d}{d\tau}\tilde I^\gamma(\tau)\Big| \le \alpha + \beta.$$

Thus there exists $C>0$ depending only on $f_0, \rho, \sigma, \eta, \alpha, \beta$ such that for any $s,t \ge 0$,

$$\begin{aligned} &|\tilde W^\gamma(t) - \tilde W^\gamma(s)| \le C|t-s|,\\ &|\tilde I^\gamma(t) - \tilde W^\gamma(s)| \le C|t-s|. \end{aligned} \tag{D.6}$$

We endow $\tilde X := P([0,R]) \times [0,R] \times [0,1]$ with the norm $\|(f,w,i)\|_{\tilde X} := \|f\|_X + w + i$. It follows from [17][Lemma 5.3 and Corollary 5.5] that the norm $\|.\|$ defined in (D.3) induces the weak topology on $P([0,R]$. Thus $(\tilde X, \|.\|_{\tilde X})$ is compact. Moreover (D.4) and

(D.6) shows that the bounded sequence $(g^\gamma, \tilde{W}^\gamma, \tilde{I}^\gamma)_\gamma$ is uniformly equicontinuous. Hence, the Arzela-Ascoli Theorem, together with a diagonal argument, ensure the existence of (g, W, I) with $g \in C([0,\infty); P([0,R]))$, $W \in C([0,\infty); [0,R])$, $I \in C([0,\infty); [0,1])$ and a subsequence $(\gamma_n)_n$ converging to 0 such that $(g^{\gamma_n}, \tilde{W}^{\gamma_n}, \tilde{I}^{\gamma_n})_n$ converges to (g, W, I) in $C([0,T]; P([0,R])) \times C([0,T]; [0,R]) \times C([0,T]; [0,1])$ for any $T > 0$.

It remains to show that (g, W, I) is a solution with initial conditions (f_0, W_0, I_0). For ease of notation we still write γ instead of γ_n. Given $\tau \geq 0$ and $\phi \in C^1([0,R])$, we have

$$\begin{aligned}\int_0^R \phi\, dg^\gamma_\tau - \int \phi\, df_0 &= \int_0^\tau \int_0^R \phi'(a)(\tilde{W}^\gamma(t) - a)\, dg^\gamma_t(a)dt + O(\gamma),\\ \tilde{W}^\gamma(\tau) - W_0 &= \int_0^\tau \tilde{W}^\gamma(t)\Big(\rho \tilde{I}^\gamma(t) + \sigma(a^*_\delta - \langle a\rangle) + \eta(a^*_\delta - \tilde{W}^\gamma(t))\Big)\, dt,\\ \tilde{I}^\gamma(\tau) - I_0 &= \int_0^\tau \alpha\langle e^{-a}\rangle \tilde{I}^\gamma(t)(1 - \tilde{I}^\gamma(t)) - \beta \tilde{I}^\gamma(t)\, dt.\end{aligned}$$

Since a and e^{-a} are continuous function on $[0,R]$, we have $\langle a\rangle = \int_0^R a\, dg^\gamma_\tau \to \int_0^R a\, dg_\tau$ and $\langle e^{-a}\rangle = \int_0^R e^{-a}\, dg^\gamma_\tau \to \int_0^R e^{-a}\, dg_\tau$ as $\gamma \to 0$. We can then pass to the limit $\gamma \to 0$ in the equation for $\tilde{W}^\gamma$ and $\tilde{I}^\gamma$. Moreover for any $t \geq 0$,

$$\begin{aligned}\int_0^R \phi'(a)(\tilde{W}^\gamma(t) - a)\, dg^\gamma_t(a) &= \int_0^R \phi'(a)(W(t) - a)\, dg^\gamma_t(a) + (\tilde{W}^\gamma(t) - W(t)) \int_0^R \phi'(a)\, dg^\gamma_t(a)\\ &\to \int_0^R \phi'(a)(W(t) - a)\, dg_t(a)\end{aligned}$$

with

$$\Big| \int_0^R \phi'(a)(\tilde{W}^\gamma(t) - a)\, dg^\gamma_t(a) \Big| \leq \|\phi'\|_\infty R^2.$$

We can thus pass to the limit using the Dominated Convergence Theorem in the equation for g^γ. We eventually obtain

$$\begin{aligned}\int_0^R \phi\, dg_\tau - \int \phi\, df_0 &= \int_0^\tau \int_0^R \phi'(a)(W(t) - a)\, dg_t(a)dt,\\ W(\tau) - W_0 &= \int_0^\tau W(t)\Big(\rho I(t) + \sigma(a^* - \langle a\rangle) + \eta(a^* - W(t))\Big)\, dt,\\ I(\tau) - I_0 &= \int_0^\tau \alpha\langle e^{-a}\rangle I(t)(1 - I(t)) - \beta I(t).\end{aligned}$$

This concludes the proof of Theorem 3.3.

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